

NATIONAL ADVISORY COMMITTEE FOR AERONAUTICS

TECHNICAL NOTE 2341

A LEAST SQUARES CURVE FITTING METHOD WITH APPLICATIONS
TO THE CALCULATION OF STABILITY COEFFICIENTS
FROM TRANSIENT-RESPONSE DATA

By Marvin Shinbrot

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Page	Line	In Place of	Read
4	2	$\lambda_i \neq \lambda_j, i \neq j$	$\lambda_i \neq \lambda_j \text{ if } i \neq j$
4	27	A_i	A_j
5	4	equation (6)	equation (5)
7	24	$m-1$	$m+1$
8	2	$m-1$	$m+1$
8	4	$m-1$	$m+1$
8	10	equation (1)	equation (2)
8	11	$(D^2 - a_1 D - a_0) q(t) = (C_1 D - C_0) F(t)$	$(D^2 + a_1 D + a_0) q(t) = (C_1 D + C_0) F(t)$
10	6	(5a)	(4a)
11	19	$\sum_{i=0}^{n-1} [\Delta q(t_i) - \Delta q_m(t_i)]$	$\sum_{i=0}^{n-1} [\Delta q(t_i) - \Delta q_m(t_i)]^2$
13	4	$a_1 = \lambda_1 + \lambda_2 = 2l$	$a_1 = -(\lambda_1 + \lambda_2) = -2l$
14	1	$\beta = \frac{(l'\sigma - l\sigma')C_1 - \sigma'C_0 + l'q_0}{2l'}$	$\beta = \frac{(l'\sigma - l\sigma')C_1 - \sigma'C_0 + l'q_0}{2l'}$
14	16	$(D^2 + bD + k)$	$(D^2 + a_1 D + a_0)$
14	19	$q = \dots (4a)$	$q = \dots (4b)$
16	$\left\{ \begin{array}{l} 1 \\ 2 \end{array} \right.$	$\int_0^T e^{-\lambda_1 \tau} F(\tau) d\tau$ is constant and since $F(t)$ is constant for $t \geq T$, and $\int_T^t e^{-\lambda_1 \tau} F(\tau) d\tau$ may be evaluated.	$\int_0^T e^{-\lambda_1 \tau} F(\tau) d\tau$ is constant. Further, since $F(t)$ is constant for $t \geq T$, $\int_T^t e^{-\lambda_1 \tau} F(\tau) d\tau$ may be evaluated.
17	22	F_0	$F(0)$
19	24	$\beta' \approx 2.6$	$\beta' \approx -2.6$
22	6	$\lambda l = l - l'1$	$\lambda_2 = l - l'1$
24	15	$2k\beta$	$-2a_0\beta$
24	16	$-\frac{2kl'\beta' + C_0 l}{k}$	$-\frac{2a_0 l'\beta' + C_0 l}{a_0}$
26	$\left\{ \begin{array}{l} 17 \\ 19 \end{array} \right.$	$\int_0^t e^{-\lambda_1 \tau} F(\tau) d\tau$	$\int_0^t e^{-\lambda_1 \tau} F(\tau) d\tau$

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Page	Line	In Place of	Read
26	{ 17 19	$\int_0^t e^{\lambda_2 \tau} F(\tau) d\tau$	$\int_0^t e^{-\lambda_2 \tau} F(\tau) d\tau$
28	7	$\left(\frac{\partial q}{\partial \beta'} \right)_0 e^{-\lambda_0 t}$	$\frac{(\partial q / \partial \beta')_0}{e^{\lambda_0 t}}$
30	1	$\Sigma(\ddot{q} + b\dot{q} + kq - C_1\dot{F} - C_0\ddot{F})^2$	$\Sigma(\ddot{q} + a_1\dot{q} + a_0q - C_1\dot{F} - C_0\ddot{F})^2$
31	24	$\sum_{i=1}^n B_i e^{\lambda_i \Delta t} \left[\left(e^{\lambda_i \Delta t} \right)^n + \right.$ $\left. Q_{n-1} \left(e^{\lambda_i \Delta t} \right)^{n-1} + \dots + Q_0 \right]$	$\sum_{i=1}^n B_i e^{\lambda_i k \Delta t} \left[\left(e^{\lambda_i \Delta t} \right)^n + \right.$ $\left. Q_{n-1} \left(e^{\lambda_i \Delta t} \right)^{n-1} + \dots + Q_0 \right]$
32	20	$\sum_{k=0}^{v-1} \left(q_k - \sum_{i=1}^n B_i e^{\lambda_i t k} \right)_1$	$\sum_{k=0}^{v-1} \left(q_k - \sum_{i=1}^n B_i e^{\lambda_i t k} \right)^2$
33	21	equation (2)	equation (23)
33	24	$i=0, 1, 2, 3$	$i=0, 1, 2$
33	26	equation (4)	equation (25)
37	3	$k[\theta(t) - \theta(0-)]$	$a_0[\theta(t) - \theta(0-)]$
37	3	equation not numbered	equation (27)
37	8	$\int_{-\epsilon}^t \theta(t) dt$	$\int_{-\epsilon}^{\epsilon} \theta(t) dt$
37	11	equation (19)	equation (27)
37	14	equation (4a)	equation (4b)
37	14	$q(0+)$	$\dot{q}(0+)$
37	15	$q(t) = \frac{\lambda_1 q(0-) - \lambda_1 \lambda_2 q(0) + C_1 \lambda_1 + C_0}{\lambda_1 (\lambda_1 - \lambda_2)} e^{\lambda_1 t} +$ $\frac{\lambda_2 q(0-) - \lambda_1 \lambda_2 q(0) + C_1 \lambda_2 + C_0}{\lambda_2 (\lambda_2 - \lambda_1)} e^{\lambda_2 t} +$ $\frac{C_0}{\lambda_1 \lambda_2}$	$q(t) = \frac{\lambda_1 \dot{q}(0-) - \lambda_1 \lambda_2 q(0) + C_1 \lambda_1 + C_0}{\lambda_1 (\lambda_1 - \lambda_2)} e^{\lambda_1 t} +$ $\frac{\lambda_2 \dot{q}(0-) - \lambda_1 \lambda_2 q(0) + C_1 \lambda_2 + C_0}{\lambda_2 (\lambda_2 - \lambda_1)} e^{\lambda_2 t} +$ $\frac{C_0}{\lambda_1 \lambda_2}$

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SUMMARY

The problem of calculating airplane stability parameters from the aircraft response to an arbitrary disturbance is considered. To calculate the coefficients of the linear differential equation which describes the airplane transient response, application is made of a classical least-squares curve fitting method.

It is shown that the method is applicable, although somewhat cumbersome, when the input is an arbitrary function of time. Certain inputs are demonstrated to lead to simplification in the application of the method. Examples are given illustrating the means of using the method and showing its practicability.

Finally, an appendix, in which Prony's method and a generalization thereof appears, is presented.

INTRODUCTION

The determination of the stability and control parameters of a dynamical system from its measured response has been a subject of increasing importance and interest, both from the standpoint of basic aerodynamic research and automatic stabilization of airplanes. This problem is essentially one of curve fitting and may be stated as follows: Given the time history of an airplane response to a known transient disturbance, and assuming a certain form of linear differential equation with constant coefficients describing the relation between response and disturbance, required to find the coefficients of this differential equation such that the sum of the squares of the differences between the given response and the one corresponding to the differential equation is a minimum.

A number of methods for the solution of this problem have been advanced by others and are noted in reference 1. However, these methods do not apply the least squares principle in the correct sense described above.

The method presented herein involves obtaining the solution of the differential equation in a form amenable to rigorous application of the method of least squares and such that a completely arbitrary input may be treated. A number of means are available for the fitting of a function in this form by least squares. The solution by means of a Taylor's series expansion seemingly gives good results and is used throughout this paper.

Considerable simplification of the method is possible when free oscillation data are available or when the input can be exactly or at least closely represented by a function the analytical form of which is known. Solutions for a pulse, a step, or a ramp input are given in detail or are indicated. The general solution for an arbitrary input is also included.

Selected examples of different inputs applied to the same physical system (airplane) are presented, which illustrate the different means of applying the method.

The report has been so organized that the engineer who is not interested in the derivation of the formulas used may read the first section of the report entitled "Statement of the Problem" and proceed from there directly to the section on examples.

METHOD OF ANALYSIS

Statement of the Problem

Consider a quantity $q_m(t)$ which has been measured at a set $\{t_i\}$, $i = 0, 1, \dots, v-1$ of v values of t . Suppose the initial conditions on q_m , obtained from the experimental data, are

$$\left. \begin{aligned} q_m(0) &= q_0 \\ \left(\frac{dq_m}{dt} \right)_{t=0} &= \dot{q}_0 \\ &\dots \dots \dots \\ \left(\frac{d^{n-1} q_m}{dt^{n-1}} \right)_{t=0} &= q_0^{(n-1)} \end{aligned} \right\} \quad (1)$$

where n is determined below.

A physical interpretation of these quantities may be had by considering the data $q_m(t)$ as a time history of the pitching velocity of an airplane in response to an elevator deflection $F(t)$.

Consider next the differential equation

$$P_0(D)q(t) = P_1(D)F(t) \quad (2)$$

where D is the operator d/dt , $P_0 = D^n + a_{n-1}D^{n-1} + \dots + a_0$ and $P_1 = C_mD^m + C_{m-1}D^{m-1} + \dots + C_0$ are two polynomials in D , and $F(t)$ is a known forcing function. Consider the set of all possible solutions of this differential equation obtained by varying the constants a_i and C_j . Let $q_c(t)$ be that solution of equation (2) for which

$$M = \sum_{i=0}^{v-1} [q(t_i) - q_m(t_i)]^2 \quad (3)$$

is a minimum, and subject to the same initial conditions as q_m

$$\left. \begin{aligned} q_c(0) &= q_0 \\ \left(\frac{dq_c}{dt} \right)_{t=0} &= \dot{q}_0 \\ &\dots \dots \dots \\ \left(\frac{d^{n-1} q_c}{dt^{n-1}} \right)_{t=0} &= q_0^{(n-1)} \end{aligned} \right\} \quad (1a)$$

It is then desired to find those values of the a_i and the C_j which correspond to q_c . These values will be unique, since to each set of these constants there corresponds one and only one solution of the differential equation if the initial conditions are determined.

In most practical problems, m is less than n , and this assumption will be made throughout this paper. The extension of the method described herein to the case where m is equal to or greater than n should be clear.

Let $\lambda_1, \lambda_2, \dots, \lambda_n$ be the zeros of the polynomial $P_0(D)$. It is now assumed that $\lambda_i \neq \lambda_j, i \neq j$. The extension of the following solution to the case where not all the λ_i are distinct can be found in any textbook on elementary differential equations and needs not be included here (reference 2). It is well known that if

$$P_0'(D) = \frac{d}{dD} P_0(D)$$

the solution of equation (2) is

$$q(t) = \sum_{j=1}^n \left\{ e^{\lambda_j t} \left[A_j + \frac{P_1(\lambda_j)}{P_0'(\lambda_j)} \int_0^t e^{-\lambda_j \tau} F(\tau) d\tau \right] \right\} \quad (4)$$

where the constants A_j are functions of the initial conditions $q(0), \left(\frac{dq}{dt}\right)_{t=0}, \dots, \left(\frac{d^{n-1}q}{dt^{n-1}}\right)_{t=0}, F(0), \left(\frac{dF}{dt}\right)_{t=0}, \dots, \left(\frac{d^{m-1}F}{dt^{m-1}}\right)_{t=0}$

and of the constants $C_1, C_2, \dots, C_m$. The method described in this paper consists of fitting $q(t)$, by a least squares procedure, to a function of the form (4). Several methods for fitting such a function may be found in the literature. In this report the classical device of linearization and iteration by means of a Taylor's series (reference 3, p. 214) is used. Therefore, a detailed study of this linearization follows, but it must be understood that it is in no way essential to the method. Fitting by steepest descents (reference 4), for example, could be used in place of the Taylor's series iteration. The mechanics of this iteration are cumbersome in the general case. They will, therefore, be described for particular cases, thus, it is hoped, making the method clear for any special case which may arise.

Formulation of the Method When Free Oscillation Data Are Available

The method.— Suppose there exists a T such that $F(t)=0$ for all $t \geq T$. Then, for $t \geq T$, the expression

$$\left[A_1 + \frac{P_1(\lambda_j)}{P_0'(\lambda_j)} \int_0^t e^{-\lambda_j \tau} F(\tau) d\tau \right]$$

occurring in equation (4) is constant. Call this constant B_j .

Then for $t \geq T$

$$q(t) = \sum_{j=1}^n B_j e^{\lambda_j t} \quad (5)$$

By some means (see e.g., reference 1), first approximations to the constants occurring in equation (6) are found. Let $B_1^{(0)}, B_2^{(0)}, \dots, B_n^{(0)}, \lambda_1^{(0)}, \lambda_2^{(0)}, \dots, \lambda_n^{(0)}$ be the approximations to $B_1, B_2, \dots, B_n, \lambda_1, \lambda_2, \dots, \lambda_n$, respectively. The function $q(t)$ is now expanded by means of a Taylor series about the "point"

$$\left[B_1^{(0)}, \dots, B_n^{(0)}, \lambda_1^{(0)}, \dots, \lambda_n^{(0)} \right]$$

with all terms of order higher than the first omitted. From equation (5),

$$\frac{\partial q}{\partial B_i} = e^{\lambda_i t}, \quad \frac{\partial q}{\partial \lambda_i} = B_i t e^{\lambda_i t}, \quad i = 1, \dots, n$$

Denote the values of $q, \frac{\partial q}{\partial B_i}, \frac{\partial q}{\partial \lambda_i}$ at the "point" $\left[B_1^{(0)}, \dots, B_n^{(0)}, \lambda_1^{(0)}, \dots, \lambda_n^{(0)} \right]$ by $q^{(0)}, \left(\frac{\partial q}{\partial B_i} \right)^{(0)}, \left(\frac{\partial q}{\partial \lambda_i} \right)^{(0)}$, respectively.

Then by Taylor's theorem, omitting all higher order terms,

$$\Delta q = \sum_{i=1}^n \left(\frac{\partial q}{\partial B_i} \right)^{(0)} \Delta B_i + \sum_{i=1}^n \left(\frac{\partial q}{\partial \lambda_i} \right)^{(0)} \Delta \lambda_i \quad (6)$$

where

$$\Delta q = q - q^{(0)}$$

$$\Delta B_i = B_i - B_i^{(0)}$$

$$\Delta \lambda_i = \lambda_i - \lambda_i^{(0)}$$

$$i=1, \dots, n$$

Therefore, from equation (6)

$$\Delta q = \sum_{i=1}^n e^{\lambda_i^{(0)} t} \left[\Delta B_1 + B_1^{(0)} t \Delta \lambda_1 \right] \quad (7)$$

The desired equations which lead to the minimization of

$$M = \sum_{i=0}^{v-1} \left[q_c(t_i) - q_m(t_i) \right]^2 \quad \text{may now be found as follows: Let}$$

$$\Delta q_m = q_m - q^{(0)}$$

Then

$$M = \sum_{i=0}^{v-1} \left[\Delta q(t_i) - \Delta q_m(t_i) \right]^2$$

The equations for the increments then become

$$\frac{\partial M}{\partial (\Delta B_1)} = 0$$

$$\frac{\partial M}{\partial (\Delta \lambda_1)} = 0$$

$$i=1, \dots, n$$

(8)

Since the expression (7) for Δq is linear in (ΔB_1) and $(\Delta \lambda_1)$, the equations (8) will be precisely those found when fitting redundant data to a linear function by least squares (reference 3, pp. 209-211). Equations (8) are solved for the increments ΔB_1 and $\Delta \lambda_1$, and the resulting values are added to $B_1^{(0)}$ and $\lambda_1^{(0)}$, respectively, to give the second approximation.

The iteration process by means of Taylor's expansions is now repeated until two successive iterations give the same values for the parameters to the desired number of significant figures.

It is of course true that in many cases, especially those in which very good data are available, no such Taylor series expansion is necessary, Prony's method (appendix A) giving sufficiently accurate values of the parameters. Examples have been found, however, where Prony's method did not lead to sufficiently good values, and the Taylor series was essential.

Final calculation of the coefficients of $P_0(D)$ and $P_1(D)$.— All that now remains for the problem to be completely solved is the computation of the coefficients of $P_0(D)$ and $P_1(D)$ from the calculated values of B_1 and λ_1 .

Since $P_0(D) = D^n + a_{n-1} D^{n-1} + \dots + a_0$, and since $\lambda_1, \dots, \lambda_n$ are the zeros of this polynomial,

$$\left. \begin{aligned} a_{n-1} &= - \sum_{i=1}^n \lambda_i \\ a_{n-2} &= \sum_{\substack{i,j=1 \\ i < j}}^n \lambda_i \lambda_j \\ &\dots\dots\dots \\ a_0 &= (-1)^n \prod_{i=1}^n \lambda_i \end{aligned} \right\} \quad (9)$$

(reference 5, p. 29).

The constants B_1 were defined as follows:

$$B_1 = A_1 + \frac{P_1(\lambda_1)}{P_0'(\lambda_1)} \int_0^T e^{-\lambda_1 \tau} F(\tau) d\tau$$

where T is that value of t where free oscillation begins. It is well known that the A_1 may be solved for as functions of the constants $C_1, \dots, C_m$. The integrals occurring in the definition of B_1 can now be found graphically since λ_1 and $F(t)$ are known. There are, therefore, n equations in the $m-1$ unknowns $C_0, \dots, C_m$ (the

coefficients of P_1). It was assumed that m is less than n . Therefore, $m-1$ is not greater than n . If $m-1$ is equal to n , the number of equations is equal to the number of unknowns, and the constants C_i may be determined. If $m-1$ is less than n , a least square procedure for solving redundant linear equations may again be used (reference 3, pp. 209-211).

The Method for a Second-Order System

A case of great practical interest in aerodynamics is that of the second order, and in the present section this case will be treated in detail. A second-order system is one in which equation (1) becomes

$$(D^2 - a_1 D - a_0) q(t) = (C_1 D - C_0) F(t)$$

An example of such a second-order system which occurs in aerodynamics may be obtained by letting q be the response in pitching velocity to an elevator input $F(t)$. In this case (reference 1),

$$a_1 = \frac{L_{\dot{\alpha}}}{mV_0} - \frac{M_{\dot{\alpha}}}{I_y} - \frac{M_q}{I_y}$$

$$a_0 = -\frac{L_{\alpha}}{mV_0} \times \frac{M_q}{I_y} - \frac{M_{\alpha}}{I_y}$$

$$C_0 = \frac{M_{\delta}}{I_y} \times \frac{L_{\alpha}}{mV_0} - \frac{M_{\alpha}}{I_y} \times \frac{L_{\delta}}{mV_0}$$

$$C_1 = \frac{M_{\delta}}{I_y} - \frac{L_{\delta}}{mV_0} \times \frac{M_{\dot{\alpha}}}{I_y}$$

where

m mass of airplane

V_0 trim velocity of airplane

I_y pitching moment of inertia

L lift force
 M pitching moment
 δ elevator input $F(t)$
 α angle of attack

$$\dot{\alpha} \quad \frac{d\alpha}{dt}$$

$$L_{\delta} \quad \frac{\partial L}{\partial \delta}$$

$$L_{\alpha} \quad \frac{\partial L}{\partial \alpha}$$

$$M_{\delta} \quad \frac{\partial M}{\partial \delta}$$

$$M_{\alpha} \quad \frac{\partial M}{\partial \alpha}$$

$$M_{\dot{\alpha}} \quad \frac{\partial M}{\partial \dot{\alpha}}$$

$$M_q \quad \frac{\partial M}{\partial q}$$

The general solution of $(D^2 + a_1D + a_0)q = (C_1D + C_0)F$ may be obtained from equation (4) which becomes, if λ_1 and λ_2 are the roots of $x^2 + a_1x + a_0 = 0$,

$$q = A_1 e^{\lambda_1 t} + A_2 e^{\lambda_2 t} + \frac{C_1 \lambda_1 + C_0}{\lambda_1 - \lambda_2} e^{\lambda_1 t} \int_0^t e^{-\lambda_1 \tau} F(\tau) d\tau +$$

$$\frac{C_1 \lambda_2 + C_0}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \int_0^t e^{-\lambda_2 \tau} F(\tau) d\tau$$

Differentiating, an expression for $\dot{q} = \frac{dq}{dt}$ may be found, and letting $t=0$, it may be seen that

$$A_1 + A_2 = q(0)$$

$$A_1 \lambda_1 + A_2 \lambda_2 = \dot{q}(0) - C_1 F(0)$$

or, using equation (1a),

$$A_1 = \frac{\dot{q}_0 - q_0 \lambda_2 - C_1 F_0}{\lambda_1 - \lambda_2}$$

$$A_2 = \frac{\dot{q}_0 - q_0 \lambda_1 - C_1 F_0}{\lambda_2 - \lambda_1}$$

Therefore,

$$q = \frac{\dot{q}_0 - q_0 \lambda_2 - C_1 F_0 + (C_1 \lambda_1 + C_0) \int_0^t e^{-\lambda_1 \tau} F(\tau) d\tau}{\lambda_1 - \lambda_2} e^{\lambda_1 t} + \frac{\dot{q}_0 - q_0 \lambda_1 - C_1 F_0 + (C_1 \lambda_2 + C_0) \int_0^t e^{-\lambda_2 \tau} F(\tau) d\tau}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \quad (5a)$$

Pulse input.— When a pulse-type input has been applied to a second-order system, that is, when there is a T such that $F(t)=0$ for all $t \geq T$,

$$\left. \begin{aligned} B_1 &= A_1 + \frac{C_1 \lambda_1 + C_0}{\lambda_1 - \lambda_2} \int_0^T e^{-\lambda_1 \tau} F(\tau) d\tau \\ B_2 &= A_2 + \frac{C_1 \lambda_2 + C_0}{\lambda_2 - \lambda_1} \int_0^T e^{-\lambda_2 \tau} F(\tau) d\tau \end{aligned} \right\} \quad (10)$$

using the same notation as before. Then, for $t \geq T$,

$$q = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t}$$

Suppose now, as is generally the case, that λ_1 and λ_2 are conjugate complex numbers. The form previously used for the description of the general method may still be followed in this case; however, simplifications occur if this notation is abandoned. If λ_1 and λ_2 are conjugate complex, B_1 and B_2 must be so also, since q is real. Let

$$\left. \begin{aligned} \lambda_1 &= \lambda + \lambda'1 \\ B_1 &= \beta + \beta'1 \end{aligned} \right\} (11)$$

Then

$$\lambda_2 = \lambda - \lambda'1, B_2 = \beta - \beta'1$$

and

$$q = 2e^{\lambda t} (\beta \cos \lambda' t - \beta' \sin \lambda' t) \quad (12)$$

It is assumed that at this point, first approximations $\lambda_0, \lambda_0', \beta_0, \beta_0'$ to the constants $\lambda, \lambda', \beta, \beta'$, respectively, have been obtained. (See appendix A for one method for finding such approximations.)

The formulas needed for the Taylor's series iteration will now be derived. From equation (12),

$$\left. \begin{aligned} \frac{\partial q}{\partial \lambda} &= 2te^{\lambda t} (\beta \cos \lambda' t - \beta' \sin \lambda' t) \\ \frac{\partial q}{\partial \lambda'} &= -2te^{\lambda t} (\beta \sin \lambda' t + \beta' \cos \lambda' t) \\ \frac{\partial q}{\partial \beta} &= 2e^{\lambda t} \cos \lambda' t \\ \frac{\partial q}{\partial \beta'} &= -2e^{\lambda t} \sin \lambda' t \end{aligned} \right\} (13)$$

Using a similar notation to that used in the previous section on the Taylor's series expansion,

$$\begin{aligned} M &= \sum_{i=0}^{v-1} \left[\Delta q(t_i) - \Delta q_m(t_i) \right] \\ &= \sum_{i=0}^{v-1} \left[\left(\frac{\partial q}{\partial \lambda} \right)^{(o)} \Delta \lambda + \left(\frac{\partial q}{\partial \lambda'} \right)^{(o)} \Delta \lambda' + \left(\frac{\partial q}{\partial \beta} \right)^{(o)} \Delta \beta + \left(\frac{\partial q}{\partial \beta'} \right)^{(o)} \Delta \beta' + q^{(o)} - q_m \right]^2 \end{aligned}$$

Taking the derivatives of M with respect to $\Delta \lambda, \Delta \lambda', \Delta \beta, \Delta \beta'$ and setting them equal to zero will result in the equations, the solution of which will give those values of the increments which minimize M . These equations thus become

$$\begin{aligned}
& \Delta l \sum \left[\left(\frac{\partial q}{\partial l} \right)^{(o)} \right]^2 + \Delta l' \sum \left(\frac{\partial q}{\partial l} \right)^{(o)} \left(\frac{\partial q}{\partial l'} \right)^{(o)} + \Delta \beta \sum \left(\frac{\partial q}{\partial l} \right)^{(o)} \left(\frac{\partial q}{\partial \beta} \right)^{(o)} + \\
& \quad \Delta \beta' \sum \left(\frac{\partial q}{\partial l} \right)^{(o)} \left(\frac{\partial q}{\partial \beta'} \right)^{(o)} = \sum \left[q_m - q^{(o)} \right] \left(\frac{\partial q}{\partial l} \right)^{(o)} \\
& \Delta l \sum \left(\frac{\partial q}{\partial l'} \right)^{(o)} \left(\frac{\partial q}{\partial l} \right)^{(o)} + \Delta l' \sum \left[\left(\frac{\partial q}{\partial l'} \right)^{(o)} \right]^2 + \Delta \beta \sum \left(\frac{\partial q}{\partial l'} \right)^{(o)} \left(\frac{\partial q}{\partial \beta} \right)^{(o)} + \\
& \quad \Delta \beta' \sum \left(\frac{\partial q}{\partial l'} \right)^{(o)} \left(\frac{\partial q}{\partial \beta'} \right)^{(o)} = \sum \left[q_m - q^{(o)} \right] \left(\frac{\partial q}{\partial l'} \right)^{(o)} \\
& \Delta l \sum \left(\frac{\partial q}{\partial \beta} \right)^{(o)} \left(\frac{\partial q}{\partial l} \right)^{(o)} + \Delta l' \sum \left(\frac{\partial q}{\partial \beta} \right)^{(o)} \left(\frac{\partial q}{\partial l'} \right)^{(o)} + \Delta \beta \sum \left[\left(\frac{\partial q}{\partial \beta} \right)^{(o)} \right]^2 + \\
& \quad \Delta \beta' \sum \left(\frac{\partial q}{\partial \beta} \right)^{(o)} \left(\frac{\partial q}{\partial \beta'} \right)^{(o)} = \sum \left[q_m - q^{(o)} \right] \left(\frac{\partial q}{\partial \beta} \right)^{(o)} \\
& \Delta l \sum \left(\frac{\partial q}{\partial \beta'} \right)^{(o)} \left(\frac{\partial q}{\partial l} \right)^{(o)} + \Delta l' \sum \left(\frac{\partial q}{\partial \beta'} \right)^{(o)} \left(\frac{\partial q}{\partial l'} \right)^{(o)} + \\
& \quad \Delta \beta \sum \left(\frac{\partial q}{\partial \beta'} \right)^{(o)} \left(\frac{\partial q}{\partial \beta} \right)^{(o)} + \Delta \beta' \sum \left[\left(\frac{\partial q}{\partial \beta'} \right)^{(o)} \right]^2 \\
& \quad = \sum \left[q_m - q^{(o)} \right] \left(\frac{\partial q}{\partial \beta'} \right)^{(o)} \tag{14}
\end{aligned}$$

As before, it is now assumed that this iteration is repeated until two successive iterations give the same values for the parameters to the desired number of significant figures.

The parameters λ , λ' , β and β' have thus been determined. The solution to our problem will be complete if we can, from these values, calculate a_0 , a_1 , C_0 , C_1 . From equations (9),

$$a_1 = \lambda_1 + \lambda_2 = 2\lambda$$

$$a_2 = \lambda_1 \lambda_2 = \lambda^2 + \lambda'^2$$

A means of finding C_1 and C_0 must now be found. In order to accomplish this, consider

$$\int_0^T e^{-\lambda_1 \tau} F(\tau) d\tau = \int_0^T e^{-\lambda \tau} \cos \lambda' \tau F(\tau) d\tau$$

$$-i \int_0^T e^{-\lambda \tau} \sin \lambda' \tau F(\tau) d\tau$$

Let

$$\sigma = \int_0^T e^{-\lambda \tau} \cos \lambda' \tau F(\tau) d\tau$$

Let

$$\sigma' = \int_0^T e^{-\lambda \tau} \sin \lambda' \tau F(\tau) d\tau$$

Then

$$\int_0^T e^{-\lambda_1 \tau} F(\tau) d\tau = \sigma - \sigma' i$$

and since λ_2 is the conjugate of λ_1 ,

$$\int_0^T e^{-\lambda_2 \tau} F(\tau) d\tau = \sigma + \sigma' i$$

Recalling that $B_1 = \beta + \beta' i$, it follows from equation (4a) that

$$\left. \begin{aligned} \beta &= \frac{(\lambda' \sigma - \lambda \sigma') C_1 - \sigma' C_0 + \lambda' q_0}{2\lambda'} \\ \beta' &= \frac{\lambda q_0 - \dot{q}_0 - [\lambda \sigma + \lambda' \sigma' - F(0)] C_1 - \sigma C_0}{2\lambda'} \end{aligned} \right\} \quad (15)$$

These equations may be solved for C_0 and C_1 .

The method when the forcing function continues throughout the motion.— It is now supposed that in the given data, there is no T such that the forcing function $F(t)=0$ for all $t \geq T$. In this case, the coefficient of $e^{\lambda_j t}$ in the expression for $q(t)$ given by equation (4) is no longer constant. Prony's method cannot, therefore, be used for the first approximation. The simplification which occurs in the Taylor's expansion in the previous case due to the fact that $q(t)$ is a sum of exponentials with constant coefficients is also not present here.

There are cases, however, even when free oscillation does not occur where the present method can be applied with little difficulty. The best examples of such cases are those in which $F(t)$ is known to have a certain analytical form. As an example, the two degrees of freedom system $(D^2 + bD + k) q(t) = (C_1 D + C_0) F(t)$ may be considered, where

$$F(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases} \quad \text{the so-called "step" function.}$$

For this input, equation (4a) becomes

$$q = \frac{\lambda_1 \dot{q}_0 - \lambda_1 \lambda_2 q_0 + C_0}{\lambda_1 (\lambda_1 - \lambda_2)} e^{\lambda_1 t} + \frac{\lambda_2 \dot{q}_0 - \lambda_1 \lambda_2 q_0 + C_0}{\lambda_2 (\lambda_2 - \lambda_1)} e^{\lambda_2 t} + \frac{C_0}{\lambda_1 \lambda_2} \quad (4a)$$

where, since dq/dt is discontinuous at $t=0$ if $F(t)$ is a step, $\dot{q}_0 = \lim_{t \rightarrow 0+} \frac{dq}{dt}$, $t \rightarrow 0+$ indicating that t is to approach zero through positive values only. The apparent anomaly that q is independent of C_1 is resolved in appendix B, where it is shown that

$$\dot{q}(0+) = \dot{q}(0-) + C_1 F(0+) = \dot{q}(0-) + C_1$$

First approximations to the constants

$$B_1 = \frac{\lambda_1 \dot{q}(0+) - \lambda_1 \lambda_2 q_0 + C_0}{\lambda_1(\lambda_1 - \lambda_2)} = \frac{\lambda_1 \dot{q}(0-) - \lambda_1 \lambda_2 q_0 + C_1 \lambda_1 + C_0}{\lambda_1(\lambda_1 - \lambda_2)}$$

$$B_2 = \frac{\lambda_2 \dot{q}(0+) - \lambda_1 \lambda_2 q_0 + C_0}{\lambda_2(\lambda_2 - \lambda_1)} = \frac{\lambda_2 \dot{q}(0-) - \lambda_1 \lambda_2 q_0 + C_1 \lambda_2 + C_0}{\lambda_2(\lambda_2 - \lambda_1)}$$

$$B_3 = \frac{C_0}{\lambda_1 \lambda_2}$$

which occur in equation (4b) can be found by a simple extension to Prony's method. (See appendix A.) A Taylor's expansion may then be applied to obtain a closer approximation.

A very important input, closely related to the step is the so-called "pseudo-step." In any physical case, an exact step can never be obtained for an input. The input will always have a certain finite slope near $t=0$; there may also be a certain amount of overshoot or, reciprocally, the input may undershoot its steady state value. Any input which rapidly (but not instantaneously) attains a constant nonzero value will be called a pseudo-step. Suppose then that $F(t)$ is such a function. Then there is a value T such that $F(t)$ is a constant for all $t \geq T$. Suppose this constant is c . Let this function be applied to the second-order system $(D^2 + a_1 D + a_0) q(t) = (C_1 D + C_0) F(t)$. The response in q is then given by equation (4a). It is here assumed for simplicity that the initial conditions $q(0)$ and $\dot{q}(0)$ are both zero; $F(0)$ is also zero. Then

$$q(t) = \frac{C_1 \lambda_1 + C_0}{\lambda_1 - \lambda_2} e^{\lambda_1 t} \int_0^t e^{-\lambda_1 \tau} F(\tau) d\tau + \frac{C_1 \lambda_2 + C_0}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \int_0^t e^{-\lambda_2 \tau} F(\tau) d\tau$$

For $t \geq T$,

$$q(t) = \frac{C_1 \lambda_1 + C_0}{\lambda_1 - \lambda_2} e^{\lambda_1 t} \left[\int_0^T e^{-\lambda_1 \tau} F(\tau) d\tau + \int_T^t e^{-\lambda_1 \tau} F(\tau) d\tau \right] +$$

$$\frac{C_1 \lambda_2 + C_0}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \left[\int_0^T e^{-\lambda_2 \tau} F(\tau) d\tau + \int_T^t e^{-\lambda_2 \tau} F(\tau) d\tau \right]$$

$\int_0^T e^{-\lambda_1 \tau} F(\tau) d\tau$ is constant and since $F(t)$ is constant for $t \geq T$, and $\int_T^t e^{-\lambda_1 \tau} F(\tau) d\tau$ may be evaluated.

Let

$$\frac{c}{\lambda_1} \frac{C_1 \lambda_1 + C_0}{\lambda_1 - \lambda_2} = G_1, \quad \frac{c}{\lambda_2} \frac{C_1 \lambda_2 + C_0}{\lambda_2 - \lambda_1} = G_2,$$

$$\frac{C_1 \lambda_1 + C_0}{\lambda_1 - \lambda_2} \int_0^T e^{-\lambda_1 \tau} F(\tau) d\tau = H_1, \quad \frac{C_1 \lambda_2 + C_0}{\lambda_2 - \lambda_1} \int_0^T e^{-\lambda_2 \tau} F(\tau) d\tau = H_2$$

Then G_1, G_2, H_1, H_2 are constants, and

$$\begin{aligned} q(t) &= H_1 e^{\lambda_1 t} + G_1 \left[e^{\lambda_1 (t-T)} - 1 \right] + H_2 e^{\lambda_2 t} + G_2 \left[e^{\lambda_2 (t-T)} - 1 \right] \\ &= \left(H_1 + \frac{G_1}{e^{\lambda_1 T}} \right) e^{\lambda_1 t} + \left(H_2 + \frac{G_2}{e^{\lambda_2 T}} \right) e^{\lambda_2 t} - (G_1 + G_2) \end{aligned}$$

Thus $q(t)$ is the sum of two exponentials plus a constant. First approximations may now be found by the extension to Prony's method given in appendix A, and Taylor's series expansions may be used to improve these values.

Another simple example which may be considered is that where $F(t) = \begin{cases} 0, & t < 0 \\ t, & t \geq 0 \end{cases}$. In this case the response of the second-order system would be

$$q = \gamma_1 e^{\lambda_1 t} + \gamma_2 e^{\lambda_2 t} - (\gamma_1 \lambda_1 + \gamma_2 \lambda_2) t - (\gamma_1 + \gamma_2)$$

if

$$q(0) = \dot{q}(0) = 0$$

where

$$\gamma_1 = \frac{C_1 \lambda_1 + C_0}{\lambda_1^2 (\lambda_1 - \lambda_2)}, \quad \gamma_2 = \frac{C_1 \lambda_2 + C_0}{\lambda_2^2 (\lambda_2 - \lambda_1)}$$

A first approximation may be found by several means, the derivative method (see example 3) being only one of these. Again, Taylor's series may be applied to improve these values.

In the most general case, where $F(t)$ is only known graphically or tabularly, the method can still be applied; there is a certain amount of added labor involved, but in case such an example should occur, the method for this general case is outlined below. It is to be noted that this general case is better suited to computation with a high-speed electronic computer than it is to manual computation. However, it may certainly be applied manually.

The entire method proceeds from equation (4). It is assumed that the given differential equation has been solved for q in this form.

Step 1.— By some means, a first approximation to the parameters must be found (several such methods may be found in reference 1).

Step 2.— The function $q(t)$ is expanded in a Taylor's series, all terms of order higher than the first being omitted. Herein lies most of the computation, since $\int_0^t e^{-\lambda_1 \tau} F(\tau) d\tau$ must be found graphically as a function of t .

As an example, consider again the second-order case $(D^2 + a_1 D + a_0)$ $q(t) = (C_1 D + C_0) F(t)$, where $F(t)$ is given graphically. It is assumed for simplicity that

$$F_0 = q(0) = \dot{q}(0) = 0$$

Then, using the same notation as before, $A_1 = A_2 = 0$, and $q(t)$ is simply the particular integral:

$$q(t) = \frac{C_1 \lambda_1 + C_0}{\lambda_1 - \lambda_2} e^{\lambda_1 t} \int_0^t e^{-\lambda_1 \tau} F(\tau) d\tau + \frac{C_1 \lambda_2 + C_0}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \int_0^t e^{-\lambda_2 \tau} F(\tau) d\tau$$

The computations (especially when it comes to taking derivatives for use in the Taylor series) are greatly simplified by the substitution

$$B_1 = \frac{C_1 \lambda_1 + C_0}{\lambda_1 - \lambda_2} \quad B_2 = \frac{C_1 \lambda_2 + C_0}{\lambda_2 - \lambda_1} \quad \text{The problem is then restated as follows:}$$

To find the best fit for $q(t)$ to a function of the form

$$q(t) = B_1 e^{\lambda_1 t} \int_0^t e^{-\lambda_1 \tau} F(\tau) d\tau + B_2 e^{\lambda_2 t} \int_0^t e^{-\lambda_2 \tau} F(\tau) d\tau$$

Minimization of $M = \Sigma_1 (q_c - q_m)^2$ with respect to $\beta, \beta', \lambda, \lambda'$, (where $\beta + \beta' i = B_1, \beta - \beta' i = B_2, \lambda + \lambda' i = \lambda_1, \lambda - \lambda' i = \lambda_2$) is equivalent to minimization with respect to C_1, C_0, a_1 , and a_0 , which is the desired minimization. The method now proceeds as before, with the first approximations being found and with the application of a Taylor's series.

Another means which might be employed in this general case would be to first fit $F(t)$ to some suitable function, and then find $q(t)$ as an explicit function of t as was done above for the step and the ramp.

EXAMPLES

The engineer who is interested primarily in applications may read these examples immediately after reading the first section of the report entitled "Statement of the Problem."

Example I - Pulse Input

Consider the input shown in figure 1(a) applied to the second-order system

$$(D^2 + a_1 D + a_0)q(t) = (C_1 D + C_0)F(t) \quad (16)$$

The response is given in figure 1(b). The methods and formulas described previously can be used directly for this example. If $t \geq 0.4$,

$$q = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} \quad (5a)$$

or, if $B_1 = \beta + \beta' i, \lambda_1 = \lambda + \lambda' i$, and $t \geq 0.4$,

$$\frac{q}{2} = e^{\lambda t} (\beta \cos \lambda' t - \beta' \sin \lambda' t) \quad (12)$$

It is now assumed that by some means first approximation $l_0, l_0', \beta_0, \beta_0'$, to l, l', β, β' , respectively, have been found.¹ Prony's method (see appendix A) gives $l_0 = -0.91, l_0' = 7.02, \beta_0 = 0.3681, \beta_0' = -2.6775$.

From equation (12)

$$\left. \begin{aligned} \frac{\partial}{\partial l} \left(\frac{q}{2} \right) &= t e^{lt} (\beta \cos l't - \beta' \sin l't) \\ \frac{\partial}{\partial l'} \left(\frac{q}{2} \right) &= -t e^{lt} (\beta \sin l't + \beta' \cos l't) \\ \frac{\partial}{\partial \beta} \left(\frac{q}{2} \right) &= e^{lt} \cos l't \\ \frac{\partial}{\partial \beta'} \left(\frac{q}{2} \right) &= -e^{lt} \sin l't \end{aligned} \right\} (13)$$

¹Prony's method, for example, may be used here. Prony's method is actually applied to this example in appendix A. An ordinary period and damping analysis (described below), which is shorter than Prony's method, may also be used. The afore-mentioned period and damping analysis proceeds as follows: The minima of q lie on a curve $q = K e^{lt}$, $K = \text{constant}$. Let q_1 and q_2 be two successive minima of q . Then $\left(\frac{q_2}{q_1} \right) = e^{l\tau}$, $\tau = \text{period of } q$ (which may be found from figure 1(b)). Therefore, $l = \frac{1}{\tau} \ln \left(\frac{q_2}{q_1} \right)$. As for l' , l' is equal to $\frac{2\pi}{\tau}$. Then

$$q = 2 e^{lt} (\beta \cos l't - \beta' \sin l't)$$

Fitting at any two points t_1 and t_2

$$(\cos l't_1) \beta - (\sin l't_1) \beta' = \frac{1}{2} e^{-lt_1} q(t_1)$$

$$(\cos l't_2) \beta - (\sin l't_2) \beta' = \frac{1}{2} e^{-lt_2} q(t_2)$$

which may be solved for β and β' . This analysis gives $l \approx -0.9$, $l' \approx 7.0$. Letting $t_1 = 1.0, t_2 = 2.0$, it may be found that $\beta \approx 0.4$, $\beta' \approx 2.6$.

Therefore, by Taylor's theorem, omitting all higher order terms,

$$t(\beta \cos \lambda' t - \beta' \sin \lambda' t) \Delta \lambda - t(\beta \sin \lambda' t + \beta' \cos \lambda' t) \Delta \lambda' +$$

$$(\cos \lambda' t) \Delta \beta - (\sin \lambda' t) \Delta \beta' = \frac{q_m}{2} e^{-\lambda t} - (\beta \cos \lambda' t - \beta' \sin \lambda' t)$$

Referring to table I where circled numbers refer to columns,

$$\textcircled{17} \Delta \lambda - \textcircled{18} \Delta \lambda' + \textcircled{9} \Delta \beta - \textcircled{10} \Delta \beta' = \textcircled{19}$$

Minimizing M leads to the equations (see equations (14))

$$\Delta \lambda \Sigma \textcircled{17}^2 - \Delta \lambda' \Sigma \textcircled{17} \textcircled{18} + \Delta \beta \Sigma \textcircled{9} \textcircled{17} - \Delta \beta' \Sigma \textcircled{10} \textcircled{17} = \Sigma \textcircled{17} \textcircled{19}$$

$$-\Delta \lambda \Sigma \textcircled{17} \textcircled{18} + \Delta \lambda' \Sigma \textcircled{18}^2 - \Delta \beta \Sigma \textcircled{9} \textcircled{18} + \Delta \beta' \Sigma \textcircled{10} \textcircled{18} = - \Sigma \textcircled{18} \textcircled{19}$$

$$\Delta \lambda \Sigma \textcircled{9} \textcircled{17} - \Delta \lambda' \Sigma \textcircled{9} \textcircled{18} + \Delta \beta \Sigma \textcircled{9}^2 - \Delta \beta' \Sigma \textcircled{9} \textcircled{10} = \Sigma \textcircled{9} \textcircled{19}$$

$$-\Delta \lambda \Sigma \textcircled{10} \textcircled{17} + \Delta \lambda' \Sigma \textcircled{10} \textcircled{18} - \Delta \beta \Sigma \textcircled{9} \textcircled{10} + \Delta \beta' \Sigma \textcircled{10}^2 = - \Sigma \textcircled{10} \textcircled{19}$$

These equations give

$$\Delta \lambda = -0.0100 \quad \Delta \lambda' = 0.0052$$

$$\Delta \beta = -0.0118 \quad \Delta \beta' = -0.0320$$

or

$$\lambda = -0.9200 \quad \lambda' = 7.0252$$

$$\beta = 0.3563 \quad \beta' = -2.7095$$

A second iteration was now applied, and it was found that the increments were zero to four decimals.

Finally, for computing a_1 , a_0 , C_1 and C_0

$$a_1 = -2l = 1.84$$

$$a_0 = l^2 + l'^2 = 50.1998$$

The functions $e^{-lt} F(t) \cos l't$ and $e^{-lt} F(t) \sin l't$ are now tabulated and plotted (table II and fig. 2). These two functions must now be integrated to find σ and σ' . It is to be noted that in this example $F(t)$ is exactly equal to t for $0 \leq t \leq 0.2$, and to $(0.4 - t)$ for $0.2 \leq t \leq 0.4$, and so the integration may be done analytically. The integration may be done graphically by means of a planimeter, however, when $F(t)$ is not so clearly a perfect triangular pulse, as is usually the case. A planimeter was actually used in this example to give $\sigma = 0.00493$ and $\sigma' = 0.0405$. Equations (15) are now set up:

$$0.280 C_1 + 0.00493 C_0 = 38.0690$$

$$0.07189 C_1 - 0.0405 C_0 = 5.0061$$

which give $C_1 = 134.0$ and $C_0 = 114.2$, to four significant figures.

The true values of the parameters in the preceding example were

$$a_1 = 1.84$$

$$a_0 = 50.2$$

$$C_1 = 134.0$$

$$C_0 = 114.4$$

Example II - Step Input

Consider a "step" input $F(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$ applied to the system described by equation (16). The response is given by equation (4b).

From the response shown in figure 3 it may be seen that $q(0) = 0$, $\dot{q}(0-) = 0$. Then equation (4b) becomes

$$q(t) = \frac{C_1\lambda_1+C_0}{\lambda_1(\lambda_1-\lambda_2)} e^{\lambda_1 t} + \frac{C_1\lambda_2+C_0}{\lambda_2(\lambda_2-\lambda_1)} e^{\lambda_2 t} + \frac{C_0}{\lambda_1\lambda_2}$$

Let

$$B_1 = \frac{C_1\lambda_1+C_0}{\lambda_1(\lambda_1-\lambda_2)}$$

$$B_2 = \frac{C_1\lambda_2+C_0}{\lambda_2(\lambda_2-\lambda_1)}$$

$$B_3 = \frac{C_0}{\lambda_1\lambda_2}$$

Then, using the same notation as before, with $\lambda_1 = l + l'i$, $\lambda_2 = l - l'i$, $\lambda_1 + \lambda_2 = -a_1$, $\lambda_1\lambda_2 = a_0$,

$$B_1 = -\frac{C_0}{2a_0} - \frac{C_1a_0+C_0l}{2l'a_0} i$$

$$B_2 = -\frac{C_0}{2a_0} + \frac{C_1a_0+C_0l}{2l'a_0} i$$

Let $B_1 = \beta + \beta'i$. Then $\beta = -\frac{C_0}{2a_0}$, $\beta' = -\frac{C_1a_0+C_0l}{2l'a_0}$. But $B_3 = \frac{C_0}{a_0}$.
Therefore, $B_3 = -2\beta$. Then $\frac{q}{2} = e^{lt} (\beta \cos l't - \beta' \sin l't) - \beta$.

First approximations may now be found by the extension to Prony's method described in appendix A or by a generalization of the ordinary period and damping analysis. For simplicity, this period and damping method was used to give $l_0 = -0.91$, $l_0' = 7.0$, $\beta_0 = -1.11$, $\beta_0' = -9.7$.

The Taylor's expansion is now applied

$$\frac{q}{2} = e^{\lambda t} (\beta \cos \lambda' t - \beta' \sin \lambda' t) - \beta$$

$$\frac{\partial}{\partial \lambda} \left(\frac{q}{2} \right) = t e^{\lambda t} (\beta \cos \lambda' t - \beta' \sin \lambda' t)$$

$$\frac{\partial}{\partial \lambda'} \left(\frac{q}{2} \right) = -t e^{\lambda t} (\beta \sin \lambda' t + \beta' \cos \lambda' t)$$

$$\frac{\partial}{\partial \beta} \left(\frac{q}{2} \right) = e^{\lambda t} \cos \lambda' t - 1$$

$$\frac{\partial}{\partial \beta'} \left(\frac{q}{2} \right) = -e^{\lambda t} \sin \lambda' t$$

Therefore,

$$\begin{aligned} \frac{q}{2} = & \left[e^{\lambda t} (\beta \cos \lambda' t - \beta' \sin \lambda' t) - \beta \right] + \left[t e^{\lambda t} (\beta \cos \lambda' t - \beta' \sin \lambda' t) \right] \Delta \lambda - \\ & \left[t e^{\lambda t} (\beta \sin \lambda' t + \beta' \cos \lambda' t) \right] \Delta \lambda' + \left[e^{\lambda t} \cos \lambda' t - 1 \right] \Delta \beta - \left[e^{\lambda t} \sin \lambda' t \right] \Delta \beta' \end{aligned}$$

Or, referring to table III, $(18) \Delta \lambda + (19) \Delta \lambda' + (15) \Delta \beta - (10) \Delta \beta' = (21)$.
Again minimizing M, it may be found that

$$\Delta \lambda = -0.0100$$

$$\Delta \lambda' = 0.0242$$

$$\Delta \beta = -0.0292$$

$$\Delta \beta' = 0.3110$$

which in turn give

$$\lambda = -0.9200$$

$$\lambda' = 7.0242$$

$$\beta = -1.1392$$

$$\beta' = -9.3890$$

A second Taylor's expansion was then applied (see table IV), giving

$$\lambda = -0.9201$$

$$\lambda' = 7.0258$$

$$\beta = -1.1400$$

$$\beta' = -9.3875$$

Another iteration would lead to increments which are zero to four decimals.

For the final calculation of the coefficients,

$$a_1 = -(\lambda_1 + \lambda_2) = -2\lambda = 1.8402$$

$$a_0 = \lambda_1 \lambda_2 = \lambda^2 \lambda'^2 = 50.2084$$

$$C_0 = 2k\beta = 114.4715$$

$$C_1 = -\frac{2k\lambda'\beta' + C_0\lambda}{k} = 134.0072$$

It is to be noted that a second iteration such as was applied above was hardly important, since the results of the first Taylor's series give

$$a_1 = 1.84$$

$$a_0 = 50.19$$

$$C_0 = 114.34$$

$$C_1 = 134.00$$

the true values being, as before,

$$a_1 = 1.84$$

$$a_0 = 50.2$$

$$C_0 = 114.4$$

$$C_1 = 134.0$$

Example III - Input Requiring the General Method

Consider finally the input shown in figure 4(a) applied to the system

$$(D^2 + a_1D + a_0)q(t) = (C_1D + C_0)F(t) \quad (16)$$

Such an input might occur, for example, in a stabilized airplane where the pitching velocity q is fed back to the elevator to change the input.

First approximations by the derivative method.— The so-called "derivative method" will here be applied in order to find first approximations to the desired constants. From figure 4, F and $\dot{q}$ are found graphically as functions of t . Then $\dot{q}$ is plotted (fig. 5), and from this, $\ddot{q}$ is found and tabulated. (See table V.) Rewriting equation (16),

$$(\dot{q})a_1 + (q)a_0 - (\dot{F})C_1 - (F)C_0 = -(\ddot{q})$$

and referring to table V,

$$\textcircled{6}a_1 + \textcircled{5}a_0 - \textcircled{4}c_1 - \textcircled{3}c_0 = - \textcircled{7}$$

A least-squares analysis is now applied to this equation, giving

$$a_1 = 1.84$$

$$a_0 = 50.19$$

$$c_1 = 133.89$$

$$c_0 = 114.91$$

The high accuracy of these first approximations is due, of course, to the excellence of the data in the example. The leading objection to the derivative method obviously is the necessity of finding the derivatives graphically, which in many cases leads to errors so gross as to make the values of the parameters found in this manner entirely valueless.

The Taylor's series iteration.— Since $F(t)$ is continuous at $t=0$, the constants $q(0)$, $\dot{q}(0)$, $F(0)$ of equation (4a) may be found as usual, by inspection of figures 4 and 5. Since $q(0) = \dot{q}(0) = F(0) = 0$, from equation (4a),

$$q = \frac{C_1\lambda_1 + C_0}{\lambda_1 - \lambda_2} e^{\lambda_1 t} \int_0^t e^{\lambda_1 \tau} F(\tau) d\tau + \frac{C_1\lambda_2 + C_0}{\lambda_2 - \lambda_1} e^{\lambda_2 t} \int_0^t e^{\lambda_2 \tau} F(\tau) d\tau$$

or, letting $B_1 = \frac{C_1\lambda_1 + C_0}{\lambda_1 - \lambda_2}$, $B_2 = \frac{C_1\lambda_2 + C_0}{\lambda_2 - \lambda_1}$,

$$q = B_1 e^{\lambda_1 t} \int_0^t e^{\lambda_1 \tau} F(\tau) d\tau + B_2 e^{\lambda_2 t} \int_0^t e^{\lambda_2 \tau} F(\tau) d\tau$$

Using the same notation as before, with $\lambda_1 = l + l'i$, $B_1 = \beta + \beta'i$, $\int_0^t e^{-\lambda_1 \tau} F(\tau) d\tau = \sigma - \sigma'i$, it follows that $\lambda_2 = l - l'i$, $B_2 = \beta - \beta'i$, $\int_0^t e^{-\lambda_2 \tau} F(\tau) d\tau = \sigma + \sigma'i$, and that,

$$q = 2 e^{\lambda t} [(\beta\sigma + \beta'\sigma') \cos \lambda't + (\beta\sigma' - \beta'\sigma) \sin \lambda't] \quad (17)$$

It will also be noted that $\beta = \frac{C_1}{2}$, $\beta' = -\frac{C_1\lambda + C_0}{2\lambda'}$. Letting zero subscripts denote the values of the indicated parameters at the first approximations,

$$\lambda_0 = -0.92$$

$$\lambda'_0 = 7.02$$

$$\beta_0 = 66.945$$

$$\beta'_0 = -0.59$$

From equation (17)

$$\left. \begin{aligned} \frac{\partial q}{\partial \lambda} &= 2 e^{\lambda t} \left\{ [(\beta\sigma + \beta'\sigma')t - (\beta\rho + \beta'\rho')] \cos \lambda't + \right. \\ &\quad \left. [(\beta\sigma' - \beta'\sigma)t + (\beta'\rho - \beta\rho')] \sin \lambda't \right\} \\ \frac{\partial q}{\partial \lambda'} &= 2 e^{\lambda t} \left\{ [(\beta\sigma' - \beta'\sigma)t + (\beta'\rho - \beta\rho')] \cos \lambda't - \right. \\ &\quad \left. [(\beta\sigma + \beta'\sigma')t - (\beta\rho + \beta'\rho')] \sin \lambda't \right\} \\ \frac{\partial q}{\partial \beta} &= 2 e^{\lambda t} (\sigma \cos \lambda't + \sigma' \sin \lambda't) \\ \frac{\partial q}{\partial \beta'} &= 2 e^{\lambda t} (\sigma' \cos \lambda't - \sigma \sin \lambda't) \end{aligned} \right\} \quad (18)$$

where

$$\left\{ \begin{aligned} \rho &= \int_0^t t F(t) e^{-\lambda t} \cos \lambda't dt \\ \rho' &= \int_0^t t F(t) e^{-\lambda t} \sin \lambda't dt \end{aligned} \right.$$

The function $F(t)$ is now multiplied by the four quantities $e^{-l_0 t} \cos l'_0 t$, $e^{-l_0 t} \sin l'_0 t$, $t e^{-l_0 t} \cos l'_0 t$, $t e^{-l_0 t} \sin l'_0 t$, point by point, and the resulting products are plotted against t (fig. 6). These four curves are integrated as functions of t , giving, respectively, σ , σ' , ρ , and ρ' (table VI).

The quantities $\frac{q_0}{e^{l_0 t}}$, $\frac{(\partial q / \partial l)_0}{e^{l_0 t}}$, $\frac{(\partial q / \partial l')_0}{e^{l_0 t}}$, $\frac{(\partial q / \partial \beta)_0}{e^{l_0 t}}$, and $\left(\frac{\partial q}{\partial \beta'}\right)_0 e^{l_0 t}$ are then computed from equation (18).

Then, using a Taylor's series and table VI,

$$\textcircled{8} \Delta l + \textcircled{9} \Delta l' + \textcircled{10} \Delta \beta + \textcircled{11} \Delta \beta' = \textcircled{15}$$

Minimizing M with respect to Δl , $\Delta l'$, $\Delta \beta$, $\Delta \beta'$ gives

$$\Delta l = -0.00$$

$$\Delta l' = 0.01$$

$$\Delta \beta = 0.17$$

$$\Delta \beta' = -0.05$$

Due to the inherent inaccuracy of a planimeter, only two decimal places were preserved here.

Thus

$$l = -0.92$$

$$l' = 7.03$$

$$\beta = 67.03$$

$$\beta' = -0.64$$

The desired parameters were finally computed to be:

$$a_1 = 1.84$$

$$a_0 = 50.28$$

$$C_1 = 134.06$$

$$C_0 = 114.69$$

CONCLUDING REMARKS

A method has been described by which the coefficients of the differential equation

$$\begin{aligned} \frac{d^n q}{dt^n} + a_{n-1} \frac{d^{n-1} q}{dt^{n-1}} + \dots + a_1 \frac{dq}{dt} + a_0 q \\ = C_m \frac{d^m F}{dt^m} + C_{m-1} \frac{d^{m-1} F}{dt^{m-1}} + \dots + C_1 \frac{dF}{dt} + C_0 F \end{aligned}$$

can be calculated from the graphical knowledge of $q(t)$ and $F(t)$. It is noticed that the method may become somewhat cumbersome if the input $F(t)$ is not of certain types. The input which allows the method to be applied most easily is one which goes to zero quickly, giving free oscillation data. A simplification also occurs when the input $F(t)$ is known accurately to be a function of a certain type, such as a step, a ramp $F(t) = t$ or, perhaps, something of the form $\sum_1 A_i e^{\mu_i t}$.

The method has as its primary advantages first, the fact that the correct quantity (the sum of the squares of the differences between the calculated and the measured quantities) is minimized. The ordinary simple equations of least squares may be used in this minimization, since the equations of condition (the redundant or inconsistent equations to be solved by least squares) satisfy the restrictions under which the least squares solution is normally derived. That is, the equations are linear, and only the right-hand sides are subject to error. The so-called "derivative method" used in example III minimizes

$\Sigma(\ddot{q} + b\dot{q} + kq - C_1\dot{F} - C_0F)^2$, a quantity with little, if any meaning. Other methods, Prony's, the Laplace and Fourier methods, are subject to the same objection.

Another advantage of the method is that the data are analyzed directly in the time plane, thus eliminating the possibility of the introduction of errors due to graphical integration (in the case of Fourier and Laplace transform methods) or differentiation (derivative method). Where the inputs give free oscillation data there is a certain saving of time due to this directness, no preliminary steps having to be taken before the method can be applied.

For the usual type of problem which is found, those with the pulse- or step-type inputs, the method is at least as rapid as any other which gives comparable results.

Ames Aeronautical Laboratory,
National Advisory Committee for Aeronautics,
Moffett Field, Calif., Jan. 17, 1951.

APPENDIX A

PRONY'S METHOD

A problem which occurs quite often is that of fitting a function $q(t)$ to a sum of exponentials. One method used to fit a function of this form is known as Prony's method and is described below. (See also references 1 and 3.)

It is first assumed that the data are given in tabular form with equal intervals of the argument t . Suppose measurements of $q(t)$ are taken at instants $t_0, t_1, \dots, t_{v-1}$, $v > 2n$, where n is the number of exponentials to which q is to be fitted. Then it is assumed that $t_k = t_0 + k(\Delta t)$, $\Delta t = \text{constant}$, $k = 0, 1, \dots, v-1$. By a proper shift of the time axis, t_0 may be taken equal to zero. Let $q_k = q(t_k) = q(k\Delta t)$. Fundamental to Prony's method is the following theorem: If

$$q_k = \sum_{i=1}^n B_i e^{\lambda_i t_k} \quad (19)$$

$k = 0, 1, \dots, v-1$, then q satisfies the linear difference equation

$$q_{k+n} + Q_{n-1} q_{k+n-1} + Q_{n-2} q_{k+n-2} + \dots + Q_0 q_k = 0 \quad (20)$$

where $Q_0, \dots, Q_{n-1}$ are constants such that the roots of the equation

$$x^n + Q_{n-1} x^{n-1} + \dots + Q_0 = 0 \quad (21)$$

are $e^{\lambda_i(\Delta t)}$, $i = 1, \dots, n$. The proof is as follows:

$$\begin{aligned} q_{k+n} + Q_{n-1} q_{k+n-1} + \dots + Q_0 q_k &= \sum_{i=1}^n B_i e^{\lambda_i(k+n)\Delta t} + \\ &+ Q_{n-1} \sum_{i=1}^n B_i e^{\lambda_i(k+n-1)\Delta t} + \dots + Q_0 \sum_{i=1}^n B_i e^{\lambda_i k \Delta t} \\ &= \sum_{i=1}^n B_i e^{\lambda_i \Delta t} \left[\left(e^{\lambda_i \Delta t} \right)^n + Q_{n-1} \left(e^{\lambda_i \Delta t} \right)^{n-1} + \dots + Q_0 \right] \end{aligned}$$

Certainly there exist n constants $Q_0, \dots, Q_{n-1}$, such that $e^{\lambda_1 \Delta t}, e^{\lambda_2 \Delta t}, \dots, e^{\lambda_n \Delta t}$ are the roots of equation (21). Choosing these constants in this way then makes the bracketed expression above and, therefore, $q_{k+n} + Q_{n-1} q_{k+n-1} + \dots + Q_0 q_k$ vanish. This completes the proof.

Prony's method consists of writing down the equations

$$q_n + Q_{n-1} q_{n-1} + \dots + Q_0 q_0 = 0$$

$$q_{n+1} + Q_{n-1} q_n + \dots + Q_0 q_1 = 0$$

$$q_{n+2} + Q_{n-1} q_{n+1} + \dots + Q_0 q_2 = 0$$

.....

.....

and solving them by least squares² for $Q_i, i = 0, \dots, n-1$. From the normal equations obtained from the least-squares process, $Q_0, \dots, Q_{n-1}$ can be found. Then the roots of the equation

$$x^n + Q_{n-1} x^{n-1} + \dots + Q_0 = 0$$

are calculated, giving $e^{\lambda_i(\Delta t)}$ and, therefore, λ_i .

²Herein lies one of the objections to Prony's method, since

$$\sum_{k=0}^{v-n} (q_{k+n} + Q_{n-1} q_{k+n-1} + \dots + Q_0 q_k)^2$$

rather than

$$\sum_{k=0}^{v-1} \left(q_k - \sum_{i=1}^n B_i e^{\lambda_i t_k} \right)^2$$

is minimized, the correct minimization procedure leading, in this case, to a forbidding amount of calculation.

The coefficient, $e^{\lambda_1 t}$ of B_1 in equation (19) can be tabulated for each i , since λ_1 is known. Therefore, B_1 can be found by a second least-squares procedure. This completes Prony's method.

The objection described in the preceding footnote can be overcome by considering the λ_1 and B_1 found by Prony's method not as the best possible values of these parameters, but only as a first approximation thereof. A better approximation can then be found by means of a Taylor's series. This method is described in the body of this report.

Extension to Prony's Method

Suppose now that it is required to find some way to fit a function to a sum of exponentials plus an (unknown) constant. The solution will be presented herein for the case in which the exponentials are two in number, but the generalization to a greater number of unknowns will be evident.

To be specific, a quantity q must be fitted to a function of the form

$$q = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 \quad (22)$$

q is first fitted to a function of the form

$$q = B_1 e^{\lambda_1 t} + B_2 e^{\lambda_2 t} + B_3 e^{\lambda_3 t} \quad (23)$$

and the condition that $\lambda_3 = 0$ is put in later. Prony's method is applied first to equation (2). As before q satisfies the difference equation

$$q_{k+3} + Q_2 q_{k+2} + Q_1 q_{k+1} + Q_0 q_k = 0 \quad (24)$$

where the constants Q_i , $i=0, 1, 2, 3$, are such that the equation

$$x^3 + Q_2 x^2 + Q_1 x + Q_0 = 0 \quad (25)$$

has $e^{\lambda_1(\Delta t)}$ as roots. But $\lambda_3 = 0$. Therefore, equation (4) has unity as a root, and $1 + Q_2 + Q_1 + Q_0 = 0$. Eliminating Q_0 (say) between this and equation (24)

$$(q_{k+2} - q_k) Q_2 + (q_{k+1} - q_k) Q_1 = (q_k - q_{k+3}) \quad (26)$$

is obtained. This equation is now solved by least squares for Q_1 and Q_2 . The exponentials $e^{\lambda_1(\Delta t)}$ and $e^{\lambda_2(\Delta t)}$ are now found to be the roots of $x^2 + (Q_2 + 1)x + (Q_1 + Q_2 + 1) = 0$. Prony's method now proceeds as before.

An evident extension of this method may be used to simplify the problem of fitting a sum of exponentials when one or more of the exponents are already known.

Example of Prony's Method

Suppose the data found in column (3), table VII is to be fitted to two exponentials (see example I of the body of this report). In this case, equations (20) become

$$(5) + (4) Q_1 + (3) Q_0 = 0$$

Solving by least squares,

$$\begin{cases} Q_1 \sum (4)^2 + Q_0 \sum (3)(4) = -\sum (4)(5) \\ Q_1 \sum (3)(4) + Q_0 \sum (3)^2 = -\sum (3)(5) \end{cases}$$

which give

$$Q_0 = 0.8320$$

$$Q_1 = -1.3922$$

Equation (21) thus becomes

$$x^2 - 1.3922x + 0.8320 = 0$$

or

$$x = e^{\lambda_1(\Delta t)} = 0.6961 + 0.5895 i$$

Therefore,

$$\lambda_1 = \frac{1}{\Delta t} \ln (0.6961 + 0.5895 i)$$

$$= 10 \left[\ln \sqrt{(0.6961)^2 + (0.5895)^2} + i \arctan \frac{0.5895}{0.6961} \right]$$

$$= -0.91 + 7.02 i$$

$$\lambda_2 = -0.91 - 7.02 i$$

Since λ_1 and λ_2 are complex conjugate, so are B_1 and B_2 . Let $B_1 = \beta + \beta' i$. Then $q = 2 e^{\lambda_1 t} (\beta \cos \lambda_1' t - \beta' \sin \lambda_1' t)$. Referring to table VII, this becomes (7) β - (8) $\beta' =$ (12). Solving as before by least squares,

$$\beta = 0.3681$$

$$\beta' = -2.6775$$

APPENDIX B

DISCONTINUOUS INPUTS

Consider the differential equation

$$\frac{d^n q}{dt^n} + a_{n-1} \frac{d^{n-1} q}{dt^{n-1}} + \dots + a_1 \frac{dq}{dt} + a_0 q = C_m \frac{d^m F}{dt^m} + \dots + C_1 \frac{dF}{dt} + C_0 F \quad (2)$$

A fundamental question which often arises is the following. Suppose $F(t)$ or one of its derivatives is discontinuous at a point. At this point all higher derivatives fail to exist. What is the meaning, if any, of the differential equation (2) at this point?

An instance of this problem occurs in example II of the present report. There, $F(t)$ is a step: $F(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$. At $t = 0$ dF/dt and all higher derivatives of F do not exist.

Throughout the following discussion, let $x(0+) = \lim_{t \rightarrow 0+} x(t)$, where $x(t)$ is any function, and $t \rightarrow 0+$ indicate that t is to approach zero through positive values only. Consider again the differential equation

$$\ddot{q} + a_1 \dot{q} + a_0 q = C_1 \dot{F} + C_0 F \quad (16)$$

where dots denote differentiation with respect to t . Integrate equation (16) between the limits $-\epsilon$ and t ($\epsilon > 0$) to obtain

$$\begin{aligned} [\dot{q}(t) - \dot{q}(-\epsilon)] + a_1[q(t) - q(-\epsilon)] + a_0 \int_{-\epsilon}^t q(t) dt \\ = C_1 [F(t) - F(-\epsilon)] + C_0 \int_{-\epsilon}^t F(t) dt \end{aligned}$$

It is now assumed that $\int_{-\epsilon}^t q(t) dt$ is continuous. (In all examples used in this report, $q(t)$ represents the pitching velocity of the airplane, making $\int_{-\epsilon}^t q(t) dt$ the angle of pitch which, from physical considerations, may be seen to be continuous.)

$$\int_{-\epsilon}^t F(t)dt = \int_0^t F(t)dt = t, \text{ since } F(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

Letting $\epsilon \rightarrow 0$,

$$[\dot{q}(t) - \dot{q}(0-)] + a_1 [q(t) - q(0-)] + k[\theta(t) - \theta(0-)] = C_1 F(t) + C_0 t$$

where

$$\theta(t) = \int_0^t q(t)dt$$

Integrating once more with respect to t , this time from $-\epsilon$ to $+\epsilon$,

$$[q(\epsilon) - q(-\epsilon) - 2\epsilon \dot{q}(0-)] + a_1 [\theta(\epsilon) - \theta(-\epsilon) - 2\epsilon q(0-)] +$$

$$a_0 \left[\int_{-\epsilon}^t \theta(t)dt - 2\epsilon \theta(0-) \right] = C_1 \epsilon$$

Again letting $\epsilon \rightarrow 0$, it is seen that $q(0+) - q(0-) = 0$, since $\theta(t)$ was assumed continuous. Thus $q(t)$ is continuous at zero. Going back to equation (19) and letting $t \rightarrow 0$, $\dot{q}(0+) - \dot{q}(0-) = C_1 F(0+) = C_1$. Thus, $\dot{q}(t)$ is discontinuous at zero, and the difference between the right-hand and left-hand limits of $\dot{q}(t)$ as t approaches zero is C_1 . Thus, writing equation (4a) in terms of $\dot{q}(0-)$ rather than $q(0+)$,

$$q(t) = \frac{\lambda_1 q(0-) - \lambda_1 \lambda_2 q(0) + C_1 \lambda_1 + C_0}{\lambda_1 (\lambda_1 - \lambda_2)} e^{\lambda_1 t} + \frac{\lambda_2 q(0-) - \lambda_1 \lambda_2 q(0) + C_1 \lambda_2 + C_0}{\lambda_2 (\lambda_2 - \lambda_1)} e^{\lambda_2 t} +$$

$$\frac{C_0}{\lambda_1 \lambda_2}$$

and $q(t)$ may be seen after all to be dependent on C_1 .

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TABLE II.— FURTHER CALCULATIONS NECESSARY FOR EXAMPLE I

1	2	3	4	5	6	7	8	9	10
Row	t	0.92t	$e^{-.92t}$	7.0252t	cos (5)	sin (5)	F(t)	(4)(6)(8)	(4)(7)(8)
1	0	0	1.0000	0	1.0000	0	0	0	0
2	.05	.046	.9550	.3513	.9389	.3441	.05	.04483	.0164
3	.1	.092	.9121	.7025	.7632	.6461	.1	.06961	.0589
4	.15	.138	.8711	1.0538	.4943	.8693	.15	.06469	.1136
5	.2	.184	.8319	1.4050	.1650	.9863	.2	.02745	.1641
6	.25	.230	.7945	1.7563	-.1844	.9828	.15	-.02198	.1171
7	.3	.276	.7588	2.1075	-.5113	.8594	.1	-.03880	.0652
8	.35	.322	.7247	2.4588	-.7758	.6310	.05	-.02811	.0229
9	.4	.368	.6921	2.8100	-.9455	.3256	0	0	0



TABLE III.—TAYLOR'S SERIES ITERATION APPLIED TO EXAMPLE II

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Row	t	$q_m(t)/2$	$0.91t$	$7t$	e^t	$\cos t$	$\sin t$	6×7	6×8	1.11×9	9.7×10	1.11×10	9.7×9	0×1	12×11	13×10	12×9	11×8	10×7	9×6
1	0	0	0	0	1.00000	1.00000	0	1.00000	0	1.1100	0	0	9.70	0	-1.11	9.70	0	0	1.11	0
2	.1	5.87880	.0910	.7	.91302	.76481	.64426	.69829	.58822	.77510	5.70573	.69292	6.77341	-.30171	4.93063	7.42633	.49306	.74263	.94817	-.16183
3	.2	8.68640	.1820	1.4	.83360	.16987	.98547	.14160	.82149	.15718	7.96845	.91185	1.37392	-.85840	7.81127	2.28537	1.56225	.45707	.87513	-.23487
4	.3	7.70305	.2730	2.1	.76109	-.50498	.86313	-.38434	.68692	-.42662	6.37212	.72918	-.3.72810	-.1.38434	6.79874	-.2.99892	2.03962	-.89968	.90431	-.20569
5	.4	3.99835	.3640	2.8	.69489	-.94229	.33479	-.65479	.23264	-.72682	2.25661	.25823	-.6.35146	-.1.65479	2.98343	-.6.09323	1.19337	-.2.43729	1.01492	-.09608
6	.5	-.34015	.4550	3.5	.63445	-.93637	-.35102	-.59408	-.22270	-.65943	-.2.16019	-.24720	-.5.76298	-.1.59408	-.1.50076	-.6.00978	-.79038	-.3.00489	1.16061	.09061
7	.6	-.3.29970	.5460	4.2	.57926	-.48999	-.87173	-.28383	-.50496	-.31505	-.4.89811	-.56051	-.2.75315	-.1.28383	-.4.58306	-.3.31366	-.2.71984	-.1.98820	1.28336	.17336
8	.7	-.3.80935	.6370	4.9	.52888	-.18687	-.98239	.098832	-.51957	.10970	-.5.01983	-.57672	.95867	-.90117	-.5.14923	.38195	-.3.60467	.26737	1.34018	.23018
9	.8	-.2.05715	.7280	5.6	.48287	-.77583	-.63095	.37463	-.30467	.41584	-.2.95530	-.33818	3.63391	-.62537	-.3.37114	3.29573	-.2.69691	2.63662	1.31399	.20399
10	.9	.80590	.8190	6.3	.44087	.99985	.01728	.44080	.0076182	.48929	.073897	.0084562	4.27576	-.55920	-.4.1539	4.28422	-.37385	3.85580	1.22129	.11129
11	1.0	3.33430	.9100	7.0	.40252	.75356	.65738	.30332	.28461	.33669	2.56672	.29372	2.94220	-.69668	2.23003	3.23592	2.23003	3.23592	1.10427	-.00573
12	1.1	4.47300	1.0010	7.7	.36751	.15281	.98826	.056159	.36320	.062336	3.52304	.40315	.94474	-.94384	3.46070	.94789	3.80677	1.04268	1.01230	-.09770
13	1.2	3.95430	1.0920	8.4	.33554	-.51982	.85428	-.17442	.28665	-.19361	2.78051	.31818	-.1.69187	-.1.17442	2.97412	-.1.37369	3.56894	-.1.64843	.98018	-.12982
14	1.3	2.28475	1.1830	9.1	.30636	-.94794	.31846	-.29041	.097563	-.32236	.94636	.10829	-.2.81698	-.1.26018	1.26872	-.2.70869	1.64934	-.3.52130	1.01603	-.09397
15	1.4	.39245	1.2740	9.8	.27971	-.93016	-.36715	-.26018	-.10270	-.28880	-.99619	-.11400	-.2.52375	-.1.26018	-.70739	-.2.63775	-.99035	-.3.69285	1.09984	-.01016
16	1.5	-.85365	1.3650	10.5	.25538	-.47486	-.88006	-.12127	-.22475	-.13461	-.2.18008	-.24947	-.1.17632	-.1.12127	-.2.04547	-.1.42579	-.3.06821	-.2.13869	1.19182	.08182
17	1.6	-.1.01375	1.4560	11.2	.23317	.20381	-.97901	.047522	-.22828	.052749	-.2.21432	-.25339	.46096	-.95248	-.2.26707	.20757	-.3.62731	.33211	1.25332	.14332
18	1.7	-.20035	1.5470	11.9	.21289	.78661	-.61745	.16746	-.13145	.18588	-.1.27507	-.14591	1.62436	-.83254	-.1.46095	1.47845	-.2.48362	2.51337	1.26060	.15060
19	1.8	1.06570	1.6380	12.6	.19437	.99940	.03455	.19425	.0067155	.21562	.065140	.0074542	1.88423	-.80575	-.15048	1.89168	-.27086	3.40502	1.21618	.10618
20	1.9	2.15110	1.7290	13.3	.17746	.74209	.67030	.13169	.11895	.14618	1.15382	.13203	1.27739	-.86831	1.00764	1.40942	1.91462	2.67790	1.14346	.03346
21	2.0	2.60945	1.8200	14.0	.16203	.13572	.99075	.021991	.16053	.024410	1.55714	.17819	.21331	-.97801	1.53273	.39150	3.06546	.78300	1.67672	-.03328

$\Sigma 18^2 = 114.5777739186$
 $\Sigma 18 = 3.4142472252$
 $\Sigma 19 = -2.1151600764$
 $\Sigma 19 = 13.25873312032$
 $\Sigma 19 = -75.1204775702$
 $\Sigma 19^2 = 113.075629234$
 $\Sigma 19 = 10.0084494394$
 $\Sigma 19 = 1.75430204377$
 $\Sigma 19 = 1.8952250271$
 $\Sigma 19 = 22.4137740234$
 $\Sigma 10 = -1.06632910595$
 $\Sigma 10 = -0.0609152759$
 $\Sigma 10^2 = 2.63796281518049$
 $\Sigma 10 = -0.8789726095$



TABLE IV.—SECOND ITERATION FOR EXAMPLE II

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21
Row	t	$q_m(t)/2$	$0.9t$	$7.0242t$	$e^{-0.1t}$	$\cos t$	$\sin t$	6×7	6×6	1.1392×9	9.3890×10	1.1392×13	9.3890×14	9×9	$62 - 11$	$63 + 11$	2×18	2×19	$3 - 19$	21
1	0	0	0	0	1.0	1.0	0	1.0	0	1.1392	0	0	9.3890	0	-1.1392	9.3890	0	0	1.13920	0
2	.1	5.87880	.092	.70942	.91211	.76323	.64612	.69615	.99913	.79305	5.51322	.67136	6.53615	-0.30395	4.74017	7.20751	.14702	.72075	1.13963	-.00057
3	.2	8.68040	.184	1.40484	.83194	.16505	.98629	.13731	.82053	.15642	7.70396	.93475	1.28950	-.86669	7.94754	2.22395	1.95251	.44479	1.13986	-.00034
4	.3	7.70305	.276	2.10726	.75981	-.51129	.85941	-.38797	.62013	-.44198	6.12285	.74291	-3.64265	-.38297	6.50483	2.22395	1.96945	-.86992	1.13922	-.00098
5	.4	3.99835	.368	2.80668	.69212	-.94546	.32573	-.65437	.22544	-.74546	2.11666	.25682	-6.11388	-1.63437	2.86212	-5.88706	1.14485	-2.35482	1.13623	-.00287
6	.5	-.34015	.460	3.51210	.61128	-.93207	-.36228	-.98840	-.22870	-.67031	-2.14726	-.26054	-5.58449	-1.58840	-1.47695	-5.78903	-1.73048	-2.89252	1.13680	-.00245
7	.6	-3.89970	.552	4.21452	.57580	-.47731	-.87873	-.27484	-.96097	-.31310	-4.79055	-.57640	-2.50047	-1.27484	-4.13745	-3.15687	-2.66247	-1.89412	1.13775	-.00145
8	.7	-3.80935	.644	4.91694	.52519	-.20347	-.97908	.10666	-.91420	.12173	-4.82782	-.58578	1.00331	-.89314	-4.94955	.11753	-3.14649	.29227	1.14020	.00100
9	.8	-2.07715	.736	5.61536	.47903	.78790	-.61580	.37743	-.29492	.42997	-4.82782	-.58578	3.54369	-.62287	-3.19963	3.20764	-2.55970	2.56511	1.14248	.00328
10	.9	.80990	.828	6.32178	.43692	.99924	.03909	.13659	.012554	.49736	.29713	.30665	4.09914	-.56341	-3.35132	4.11666	-3.1619	3.70517	1.15722	.01802
11	1.0	3.33430	.920	7.02420	.39892	.73740	.67546	.29387	.26918	.33478	3.39539	.41076	2.79915	-.70613	2.19255	3.06580	2.19255	3.06580	1.14175	.00255
12	1.1	4.47300	1.012	7.72662	.36349	.12637	.99198	.049534	.36057	.092328	2.61127	.31683	-1.69443	-1.18047	2.61686	-1.37760	3.36637	-1.65312	1.13994	.00074
13	1.2	3.95430	1.104	8.42904	.33154	-.94435	.93866	-.18047	.27812	-.20559	.81920	.092328	-2.71849	-1.28924	1.14504	3.38233	1.49375	-3.40482	1.13744	.00349
14	1.3	2.28475	1.196	9.13146	.30240	-.95747	.88953	-.28954	.087251	-.32984	-.10316	-.12519	-2.71849	-1.28924	-1.14504	-2.61686	1.49375	-3.40482	1.13711	.00318
15	1.4	.39245	1.288	9.83388	.27582	-.91720	-.89672	-.11135	-.22560	-.12695	-2.11816	-.25700	-1.04947	-1.11135	-1.99131	-1.32447	-1.04100	-3.50059	1.13602	.00154
16	1.5	-.85365	1.380	10.53630	.25158	-.84560	-.86672	-.07545	-.22287	-.063152	-2.09253	-.25389	-.58048	-.94457	-2.15768	-.26659	-2.98697	-1.95371	1.13766	.00273
17	1.6	-1.01375	1.472	11.23872	.22947	-.72458	-.79038	.16982	-.12234	.19346	-1.14865	-.13937	1.59444	-.83018	-1.34211	1.55207	-3.14099	2.47562	1.14176	.00256
18	1.7	-.20035	1.564	11.94114	.20930	.81137	-.58453	.19032	.014911	.21681	.14000	.016987	1.78691	-.80968	-1.07681	1.80390	-1.3926	3.24702	1.14251	.00331
19	1.8	1.06570	1.656	12.64356	.19090	.99694	.07811	.19032	.12232	.14094	1.15034	.13977	1.16161	-.87628	1.00940	1.30118	1.91786	2.17224	1.14170	.00250
20	1.9	2.15110	1.748	13.34598	.17412	.71055	.70354	.12712	.12222	.14094	1.15034	.13977	1.16161	-.87628	1.00940	1.30118	1.91786	2.17224	1.14170	.00250
21	2.0	2.60945	1.840	14.04840	.15882	.08768	.59615	.013925	.15821	.012663	1.48543	.18023	.13074	-.96608	1.46957	.31097	2.93914	.62194	1.13988	.00668

$\Sigma 15^2 = 104.133127177$
 $\Sigma 16^2 = 2.9595744569$
 $\Sigma 17^2 = 2.2940120183$
 $\Sigma 18^2 = 12.6247133443$
 $\Sigma 19^2 = -0.0219082303$
 $\Sigma 20^2 = 103.3666471536$
 $\Sigma 21^2 = 9.549380822$

$\Sigma 14 = 1.7416536668$
 $\Sigma 15 = 0.154713270$
 $\Sigma 16 = 22.3984776563$
 $\Sigma 17 = -1.072492563$
 $\Sigma 18 = -0.0011647190$
 $\Sigma 19 = 2.61153193938$
 $\Sigma 20 = -0.00144476960$



TABLE V.—TABULATION OF $F(t)$, $\dot{F}(t)$, $q(t)$, $\dot{q}(t)$, AND $\ddot{q}(t)$ FOR EXAMPLE III

①	②	③	④	⑤	⑥	⑦
	t	F	$\dot{F}$	q	$\dot{q}$	$\ddot{q}$
1	0	0	7.03	0	0	941
2	.1	.5893	4.35	3.940	68.4	327
3	.2	.8206	.21	10.939	57.1	-532
4	.3	.6521	-3.33	12.974	-23.9	-978
5	.4	.2251	-4.81	5.847	-113.9	-702
6	.5	-.2290	-3.92	-7.837	-146.1	110
7	.6	-.5062	-1.46	-20.407	-91.3	938
8	.7	-.5141	1.23	-24.019	24.6	1266
9	.8	-.2947	2.92	-15.576	137.9	886
10	.9	.0175	3.05	1.300	184.8	5
11	1.0	.2694	1.81	18.214	138.3	-895
12	1.1	.3606	-.01	26.577	21.5	-1334
13	1.2	.2779	-1.53	22.181	-105.4	-1092
14	1.3	.0869	-2.11	7.319	-178.7	-312
15	1.4	-.1103	-1.67	-10.555	-163.6	594
16	1.5	-.2258	-.57	-22.775	-71.0	1171
17	1.6	-.2226	.60	-23.785	50.6	1155
18	1.7	-.1221	1.31	-13.711	141.4	589
19	1.8	.0152	1.32	2.053	160.2	-219
20	1.9	.1228	.75	15.764	103.1	-866
21	2.0	.1582	-.05	21.207	2.5	-1058
22	2.1	.1182	-.70	16.504	-91.2	-741
23	2.2	.0329	-.93	4.663	-134.9	-107
24	2.3	-.0525	-.71	-8.250	-112.8	520
25	2.4	-.1005	-.22	-16.219	-41.0	849
26	2.5	-.0962	.29	-16.060	42.6	755
27	2.6	-.0503	.58	-8.637	98.6	324
28	2.7	.0100	.57	1.945	104.0	-211
29	2.8	.0558	.31	10.555	61.6	-595
30	2.9	.0693	-.04	13.468	-4.3	-666
31	3.0	.0501	-.32	9.997	-61.1	-426

$$\Sigma \textcircled{6}^2 = 329048.34$$

$$\Sigma \textcircled{5}^2 = 6660.992256$$

$$\Sigma \textcircled{3} \textcircled{4} = -0.428921$$

$$\Sigma \textcircled{5} \textcircled{6} = 190.3603$$

$$\Sigma \textcircled{4} \textcircled{5} = -82.75361$$

$$\Sigma \textcircled{4} \textcircled{7} = 16887.46$$

$$\Sigma \textcircled{4} \textcircled{6} = 4691.967$$

$$\Sigma \textcircled{3} \textcircled{5} = +98.2381628$$

$$\Sigma \textcircled{3}^2 = 2.66225444$$

$$\Sigma \textcircled{3} \textcircled{6} = 83.67703$$

$$\Sigma \textcircled{5} \textcircled{7} = -334446.428$$

$$\Sigma \textcircled{3} \textcircled{7} = -4835.7037$$

$$\Sigma \textcircled{6} \textcircled{7} = 23599.0$$

$$\Sigma \textcircled{4}^2 = +159.8717$$

TABLE VI.— TAYLOR'S SERIES ITERATION APPLIED TO EXAMPLE III

1	2	3	4	5	6	7	8	9	10	11	12	13
Row	t	σ	σ'	ρ	ρ'	q_0	$\left(\frac{\partial q}{\partial t}\right)_0$	$\left(\frac{\partial q}{\partial t}\right)_0$	$\left(\frac{\partial q}{\partial \beta}\right)_0$	$\left(\frac{\partial q}{\partial \beta}\right)_0$	q_m	(12)-(7)
1	0	0	0	0	0	0	0	0	0	0	0	0
2	.1	.0297	.0149	.00191	.0029	3.9370	-.0110	-.2114	.02947	.00711	3.940	.0030
3	.2	.0692	.0884	.00755	.0138	10.9405	.5410	-.6242	.08210	.04459	10.939	-.0015
4	.3	.0526	.1813	.00287	.0309	12.9966	1.3609	-2.3545	.09799	.10445	12.974	-.0226
5	.4	.0075	.222	-.01278	.0449	5.8985	-.0912	-4.3174	.04535	.14683	5.847	-.0515
6	.5	.0094	.226	-.01141	.0572	-7.7700	-3.0022	-3.8860	-.05688	.13094	-7.837	-.0670
7	.6	.0550	.270	.01376	.0850	-20.3630	-5.9425	-1.5001	-.15168	.04685	-20.407	-.0440
8	.7	.0682	.364	.0218	.1323	-24.0440	-8.0511	3.5835	-.18022	-.07340	-24.019	.0250
9	.8	.0270	.435	-.0094	.1757	-15.6851	-5.1771	9.9342	-.11866	-.17151	-15.576	.1091
10	.9	.0001	.447	-.0317	.201	1.1454	2.3728	11.6801	.00683	-.19527	1.300	.1546
11	1.0	.0325	.465	-.0006	.235	18.0924	9.6213	7.8174	.13400	-.12840	18.214	.1216
12	1.1	.0700	.541	.0385	.303	26.5549	14.3646	-.1106	.19833	-.00064	26.577	.0221
13	1.2	.0500	.633	.0150	.383	22.2894	12.8773	-10.8027	.16760	.12714	22.181	-.1084
14	1.3	.0059	.670	-.0400	.436	7.5236	3.1852	-17.4231	.05790	.19405	7.319	-.2046
15	1.4	.0113	.674	-.0321	.473	-10.3501	-8.5878	-15.2285	-.07581	.16976	-10.555	-.2049
16	1.5	.0573	.722	.0345	.545	-22.6711	-17.0053	-6.4301	-.16872	.06866	-22.775	-.1039
17	1.6	.0670	.817	.0491	.658	-23.8430	-18.8982	6.5720	-.17860	-.05885	-23.785	.0580
18	1.7	.0243	.884	-.0217	.753	-13.9088	-10.8324	18.0802	-.10522	-.15221	-13.711	.1978
19	1.8	.0004	.894	-.0629	.805	1.8028	3.2344	20.3838	.01196	-.17032	2.053	.2502
20	1.9	.0353	.914	.0020	.871	15.5842	15.2862	13.2823	.11543	-.10984	15.764	.1798
21	2.0	.0706	.994	.0705	.999	21.1848	21.0787	.3580	.15819	-.00415	21.207	.0222
22	2.1	.0474	1.084	.0225	1.141	16.6536	17.1277	-13.8916	.12522	.09490	16.504	-.1496
23	2.2	.0044	1.117	-.0697	1.229	4.9112	4.1234	-21.3879	.03794	.14266	4.663	-.2482
24	2.3	.0135	1.122	-.0489	1.291	-8.0209	-10.1541	-18.2064	-.05883	.12177	-8.250	-.2291
25	2.4	.0594	1.174	.0591	1.413	-16.1158	-19.3557	-7.3413	-.11994	.04793	-16.219	-.1032
26	2.5	.0656	1.269	.0736	1.591	-16.1305	-20.0720	7.0345	-.12083	-.04042	-16.060	.0705
27	2.6	.0216	1.333	-.0386	1.734	-8.9181	-10.8053	18.2719	-.06694	-.10187	-8.637	.2811
28	2.7	.0010	1.342	-.0928	1.811	1.7033	3.3439	19.9773	.01174	-.11129	1.945	.2417
29	2.8	.0381	1.365	.0096	1.913	10.3942	14.8375	12.6147	.07702	-.06981	10.555	.1608
30	2.9	.0710	1.447	.1030	2.103	13.4568	19.4654	.0845	.10049	-.00133	13.468	.0112
31	3.0	.0447	1.534	.0250	2.305	10.1343	14.9646	-12.5013	.07622	.06024	9.997	-.1373

$$\Sigma (8)^2 = 4286.39139342$$

$$\Sigma (9)(11) = -35.532782573$$

$$\Sigma (8)(9) = -228.77281991$$

$$\Sigma (9)(13) = 51.02915144$$

$$\Sigma (8)(10) = 35.627944979$$

$$\Sigma (10)^2 = 0.3725473211$$

$$\Sigma (8)(11) = 0.754833829$$

$$\Sigma (10)(11) = 0.0542286301$$

$$\Sigma (8)(13) = 1.43276343$$

$$\Sigma (10)(13) = -0.014427701$$

$$\Sigma (9)^2 = 4140.34148937$$

$$\Sigma (11)^2 = 0.3777271561$$

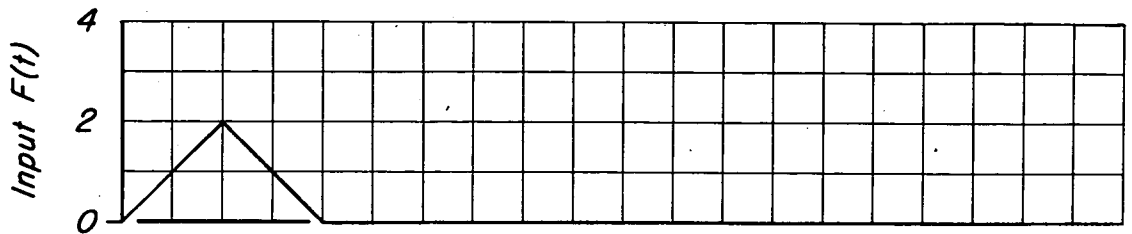
$$\Sigma (9)(10) = -4.987891922$$

$$\Sigma (11)(13) = -0.43887876$$

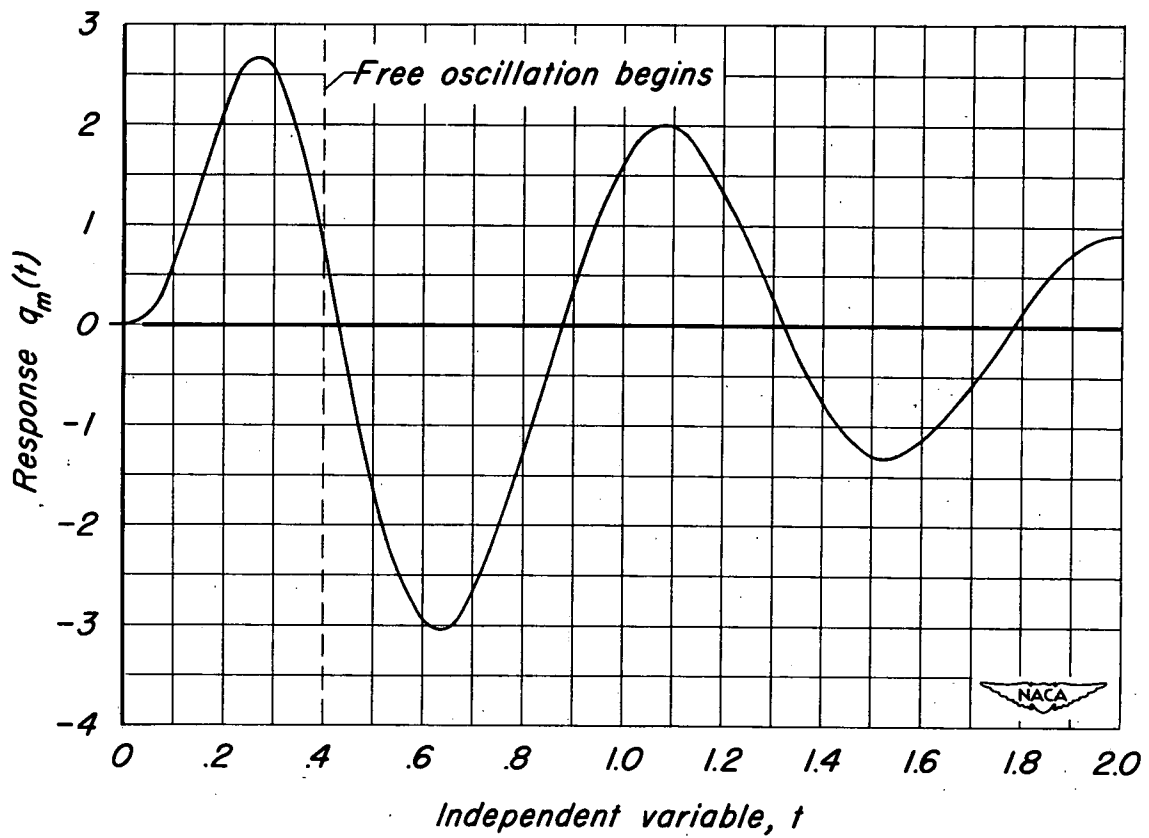
TABLE VII.—FRONY'S METHOD APPLIED TO EXAMPLE I

1	2	3	4	5	6	7	8	9	10	11	12
Row	t	q_k	q_{k+1}	q_{k+2}	$7.02t$	$\cos \textcircled{6}$	$\sin \textcircled{6}$	$1/2 \textcircled{3}$	$0.91t$	$\textcircled{10}$	$\textcircled{9} \times \textcircled{11}$
1	0.4	0.7539	-1.6600	-2.9386	2.808	-0.94495	0.32722	0.37695	0.364	1.43907	0.54246
2	.5	-1.6600	-2.9386	-2.7099	3.510	-0.93283	-0.36032	-8.3000	.455	1.57617	-1.30822
3	.6	-2.9386	-2.7099	-1.3279	4.212	-0.47946	-0.87756	-1.46930	.546	1.72633	-2.53650
4	.7	-2.7099	-1.3279	.4055	4.914	.20056	-0.97968	-1.35495	.637	1.89080	-2.56194
5	.8	-1.3279	.4055	1.6693	5.616	.78986	-0.61841	-0.66395	.728	2.07093	-1.37499
6	.9	.4055	1.6693	1.9868	6.318	.99938	.03525	.20275	.819	2.26823	.45988
7	1.0	1.6693	1.9868	1.3773	7.020	.74022	.67237	.83465	.910	2.48432	2.07354
8	1.1	1.9868	1.3773	.2644	7.722	.13105	.99138	.99340	1.001	2.72100	2.70304
9	1.2	1.3773	.2644	-.7775	8.424	-.54024	.84151	.68865	1.092	2.98023	2.05234
10	1.3	.2644	-.7775	-1.3026	9.126	-.95590	.29371	.13220	1.183	3.26415	.43152
11	1.4	-.7775	-1.3026	-1.1666	9.828	-.91955	-.39298	-.38875	1.274	3.57512	-1.39983
12	1.5	-1.3026	-1.1666	-.5405	10.530	-.44823	-.89392	-.65130	1.365	3.91572	-2.55031
13	1.6	-1.1666	-.5405	.2181	11.232	.23497	-.97200	-.58330	1.456	4.28877	-2.50164
14	1.7	-.5405	.2181	.7532	11.934	.80717	-.59032	-.27025	1.547	4.69736	-1.26946
15	1.8	.2181	.7532	.8672	12.636	.99752	.07045	.10905	1.638	5.14487	.56105

$$\begin{aligned}
 \Sigma \textcircled{4}^2 &= 33.9323 & \Sigma \textcircled{7}^2 &= 8.1685 \\
 \Sigma \textcircled{3} \textcircled{4} &= 24.7477 & \Sigma \textcircled{7} \textcircled{8} &= -1.7988 \\
 \Sigma \textcircled{4} \textcircled{5} &= 26.6524 & \Sigma \textcircled{7} \textcircled{12} &= 2.5253 \\
 \Sigma \textcircled{3}^2 &= 33.9334 & \Sigma \textcircled{8}^2 &= 6.8315 \\
 \Sigma \textcircled{3} \textcircled{5} &= 6.2232 & \Sigma \textcircled{8} \textcircled{12} &= 18.2254
 \end{aligned}$$

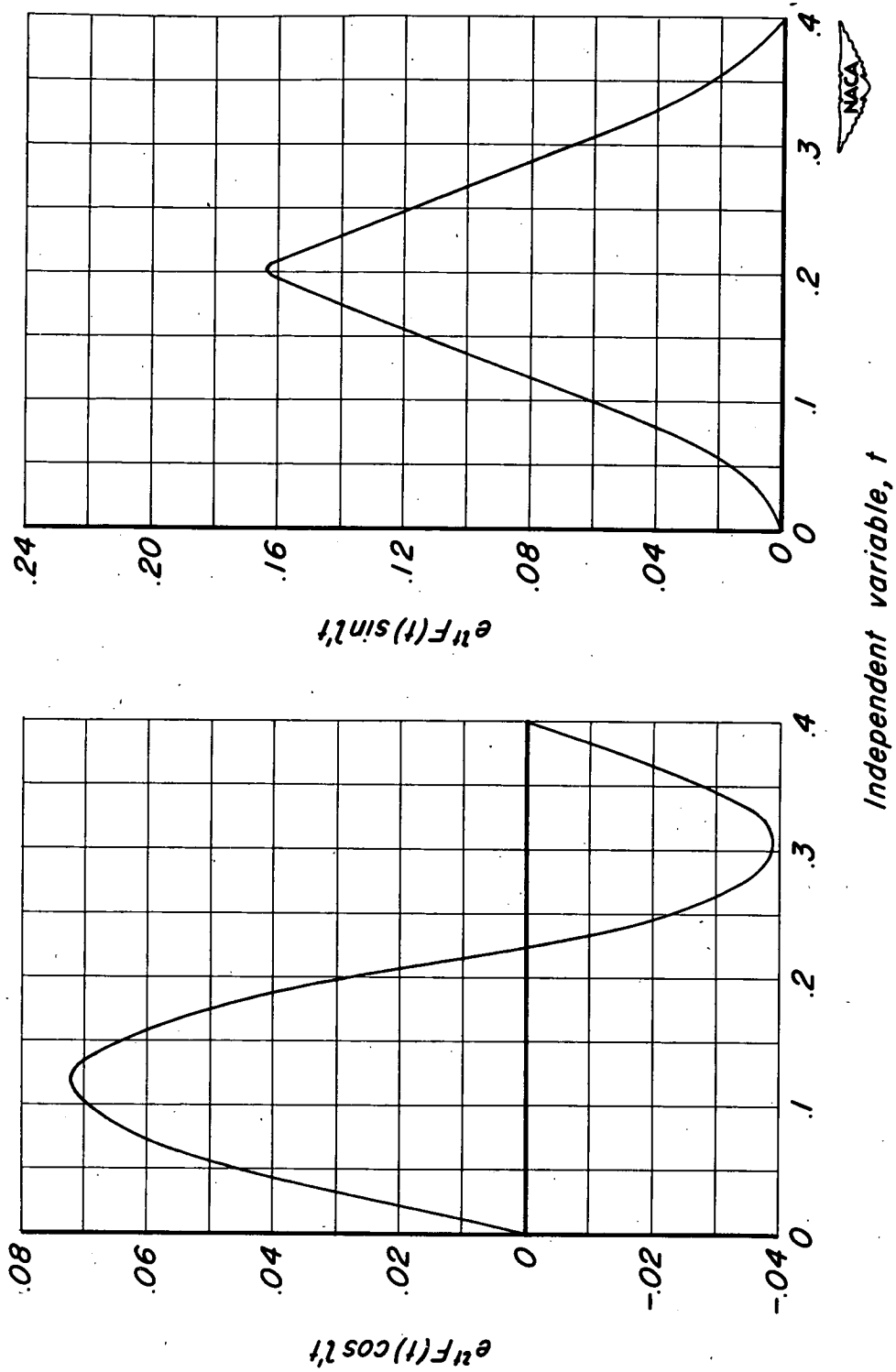


(a) Pulse input, $F(t)$.



(b) Response $q_m(t)$ to the pulse $F(t)$.

Figure 1.- Pulse input and associated response used in example I.



(a) $e''F(t)\cos \lambda't$.
 (b) $e''F(t)\sin \lambda't$.

Figure 2.—The variation with time of two quantities used in example I.

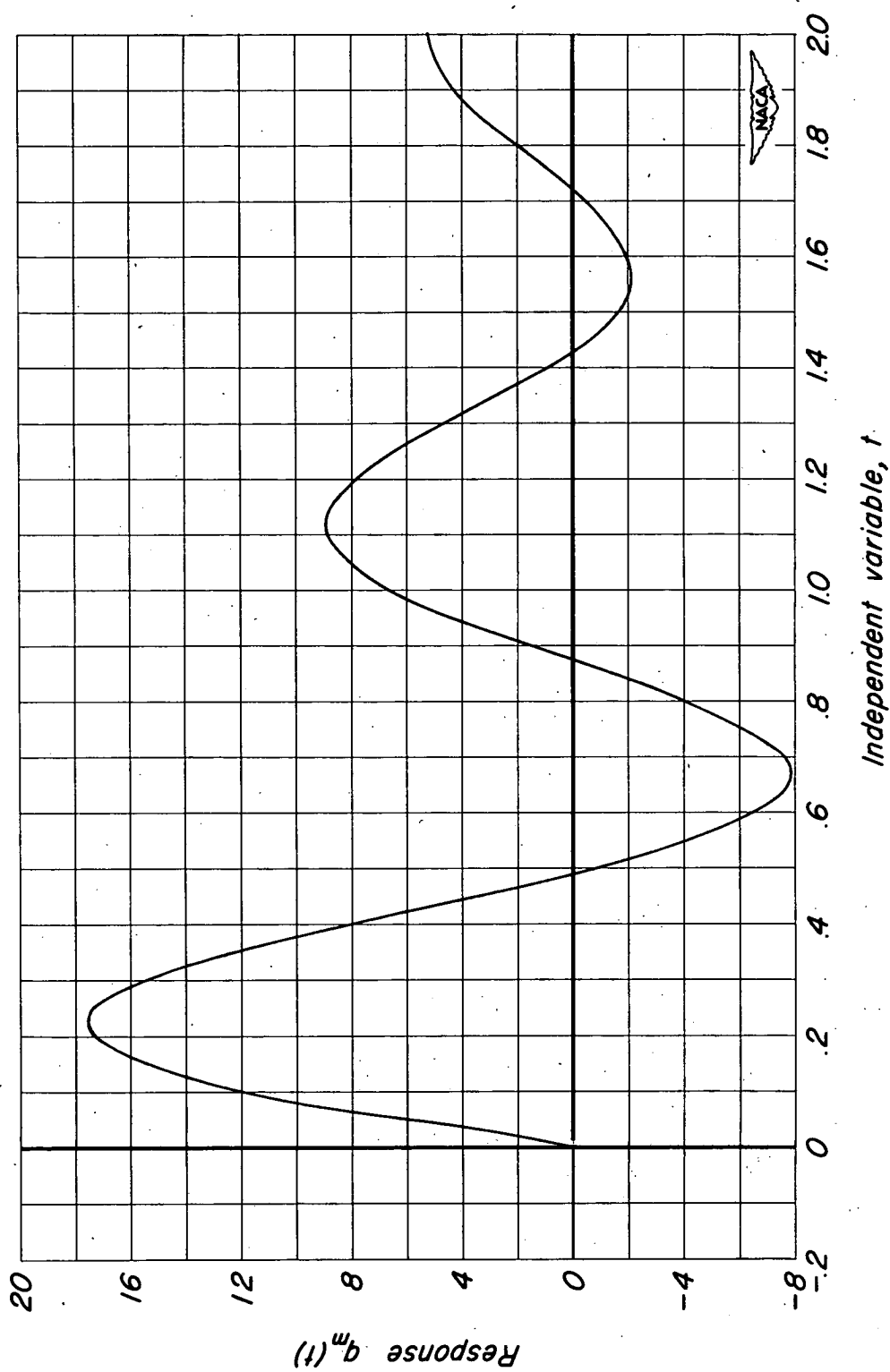
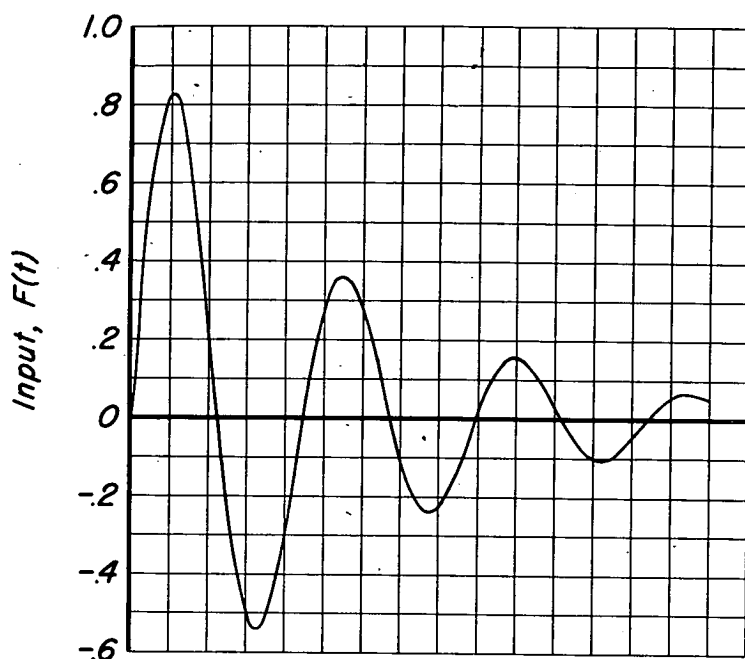
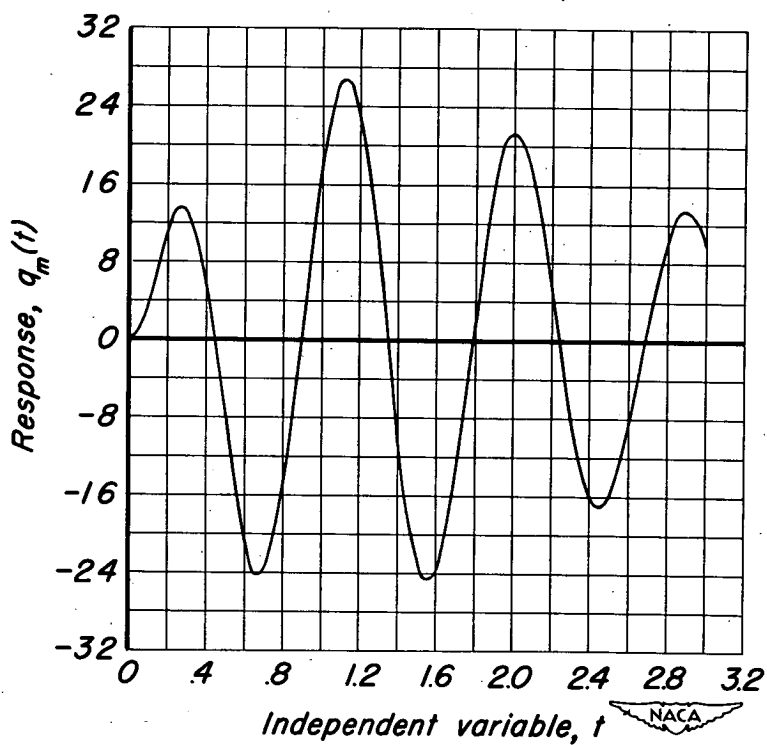


Figure 3.- Response to a step input used in example II.



(a) Input $F(t)$ used in example III.



(b) Response, $q_m(t)$, to input, $F(t)$, of figure 4(a).

Figure 4.—Input and associated response used in example III.

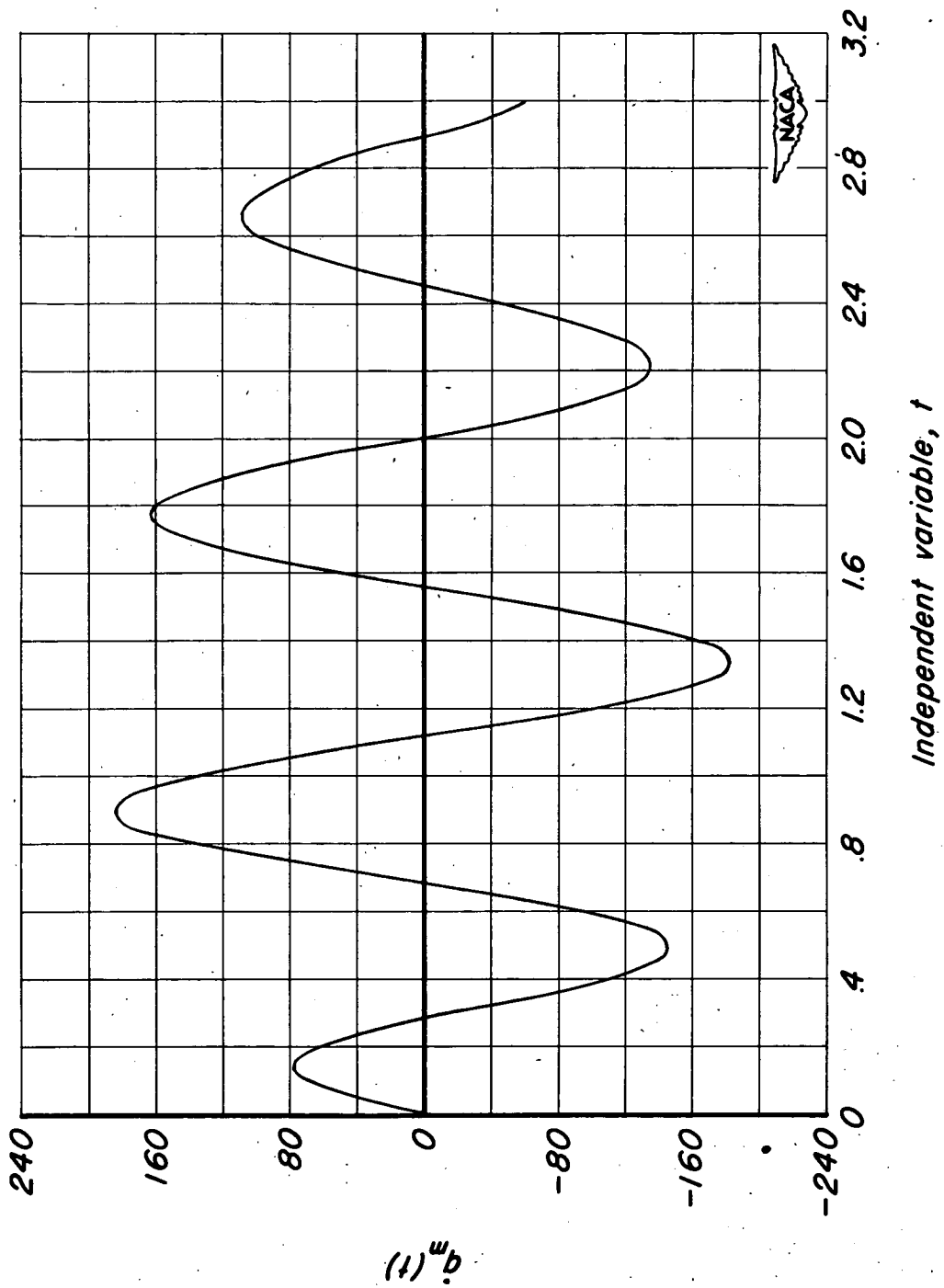


Figure 5.—First derivative, $\dot{q}_m(t)$, of the response, $q_m(t)$, used in example III.

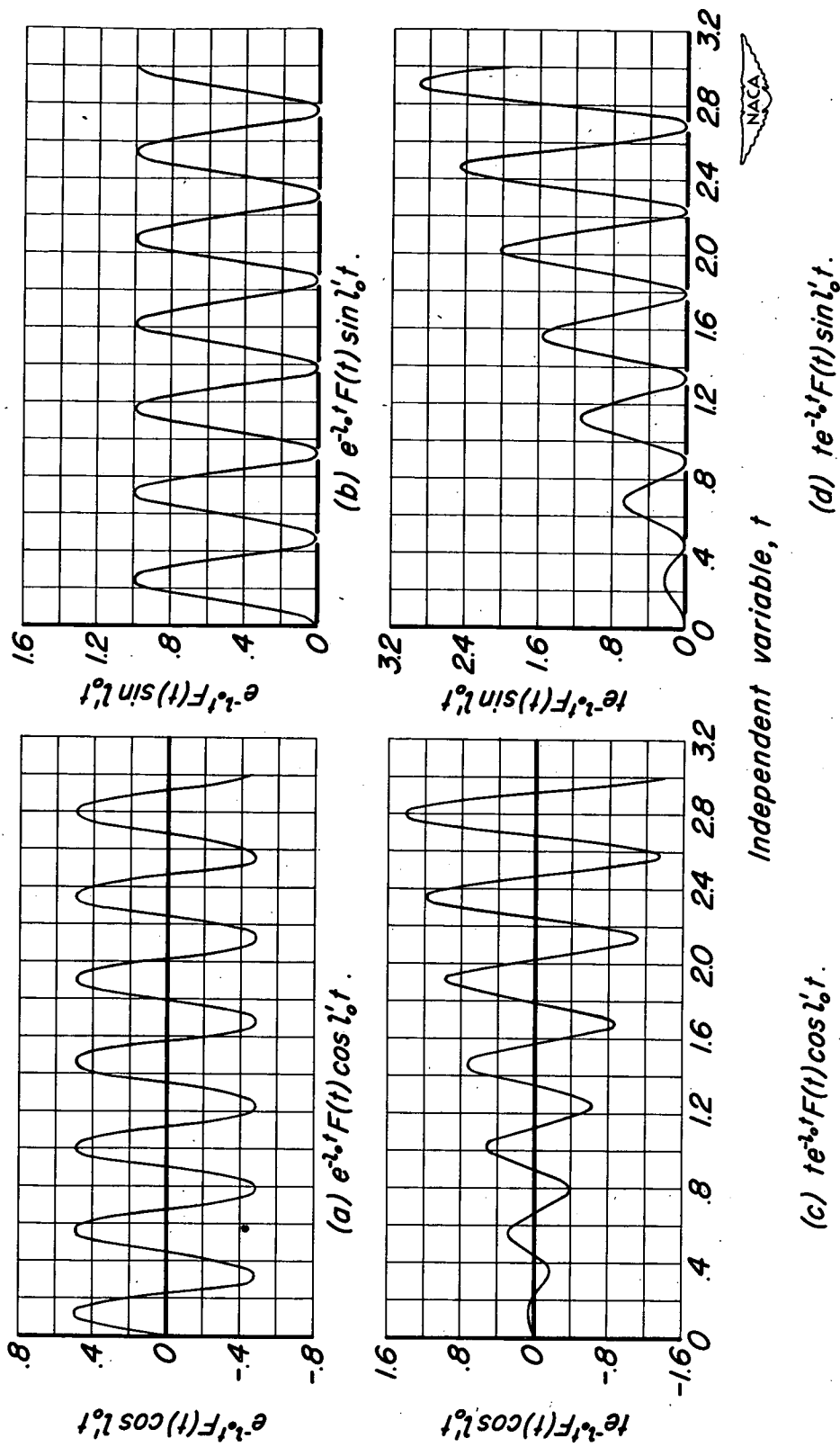


Figure 6.—The variation with time of four quantities used in example III.